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2d TQFT structure of the superconformal indices with outer-automorphism twists

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abstract

We study the superconformal indices of 4d theories coming from 6d $\mathcal{N} = (2, 0)$ theory of type Γ on a Riemann surface, with the action of the outer-automorphism σ in the trace. We find that the indices are given by the partition function of a deformed 2d Yang-Mills on the Riemann surface with gauge group G which is S-dual to the subgroup of Γ fixed by σ . In the 2-parameter deformed version, we find that it is governed *not* by Macdonald polynomials of type G , but by Macdonald polynomials associated to twisted affine root systems.

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1 Introduction

Consideration of the 6d $\mathcal{N} = (2, 0)$ theory of type A on the spacetime of the form $S^3 \times S^1 \times \mathcal{C}_2$ recently led the authors of [1, 2, 3, 4] to the following observation. The compactification of the 6d theory on \mathcal{C}_2 leads to a 4d $\mathcal{N} = 2$ theory often called a theory of class \mathcal{S} [5, 6]. In the low energy limit the theory becomes superconformal, and then its partition function on $S^3 \times S^1$ is called the superconformal index [7, 8]. This can be explicitly calculated when the theory of class \mathcal{S} in question has a Lagrangian description, and equals the partition function of deformed 2d Yang-Mills with gauge group $SU(n)$ on \mathcal{C}_2 . In the setup, we have three parameters (p, q, t) ¹. In the one-parameter subspace $(p = 0, q, t = q)$, the deformation is the standard q -deformation, and in the two-parameter subspace $(p = 0, q, t)$, the deformation is described by Macdonald polynomials of type A . The full three-dimensional parameter space corresponds to an elliptic generalization to Macdonald polynomials, about which not much is known, except some developments reported in [4].

The same setup was analyzed in [9, 10] by exchanging the order of the compactification. Let us first compactify the 6d theory on S^1 . We then have 5d maximally-supersymmetric Yang-Mills theory on $S^3 \times \mathcal{C}_2$. We can perform supersymmetric localization of this system

¹Here and in the following, we use the notation of [4] for (p, q, t) .

to show that it reduces to the 2d deformed Yang-Mills theory on \mathcal{C}_2 . Currently it is only possible to analyze the one-parameter subspace ($p = 0, q, t = q$) in this approach, where the q -deformation arises from the one-loop determinants of the Kaluza-Klein modes along S^3 .

Γ	$A_{2\ell-1}$	$D_{\ell+1}$	D_4	E_6	$A_{2\ell}$
d	2	2	3	2	2
G	B_ℓ	C_ℓ	G_2	F_4	C_ℓ
Macdonald	C_ℓ^\vee	B_ℓ^\vee	G_2^\vee	F_4^\vee	(C_ℓ^\vee, C_ℓ)

Table 1: The type of the 6d theory, the order of the outer-automorphism twists, the resulting 5d gauge group, and the type of the Macdonald polynomials governing the deformation.

A 6d $\mathcal{N} = (2, 0)$ theory is labeled by a simply-laced Dynkin diagram Γ , and the graph automorphism group of Γ is a discrete symmetry group of the theory. We call this symmetry group the outer-automorphism group. One of the mysterious features of the 6d theory is that when we compactify it on S^1 with an outer-automorphism twist σ , the resulting 5d maximally-supersymmetric Yang-Mills has the gauge group G which is *S-dual* to the subgroup invariant under σ of the simply-laced group of type Γ , see Table 1. This feature was derived using various nonperturbative dualities [11, 12], since there is still no universally accepted Lagrangian description of the 6d theory.

In this article, we extend the analysis of the superconformal index of the theory of class \mathcal{S} of type A , in the one- and two-parameter subspaces, to the case when we have an outer-automorphism twist σ around S^1 .² The class \mathcal{S} theory has sometimes a Lagrangian description, and the superconformal index can be calculated in the weakly-coupled regime. Therefore, our analysis can shed light on the mechanism how the group G S-dual to the σ -invariant subgroup arises. As we will see, the twist σ does not restrict the modes to the subspace invariant under σ . Rather, it enforces the modes exchanged by σ to appear always in pairs, effectively producing a particle whose charge is described by the group G S-dual to the σ -invariant subgroup. Thus we will find a deformed 2d Yang-Mills theory with gauge group G on \mathcal{C}_2 .

We will also find that the two-parameter version is not just obtained by considering Macdonald polynomials of type G . The Macdonald polynomials are parameterized by (possibly-non-reduced) affine root systems, which are *not* in one-to-one correspondence with non-affine Dynkin diagrams parameterizing G . The type of Macdonald polynomials which appear in the description of the 2d theory is summarized in Table 1. In particular, for $\Gamma = A_{2\ell}$ and \mathbb{Z}_2 twist, we need the polynomials associated to (C_ℓ^\vee, C_ℓ) , which are called the Koornwinder polynomials.

Any Riemann surface \mathcal{C}_2 is constructed from cylinders and three-punctured spheres. The superconformal index without the twist, in the two parameter case, is known to have the

²For $\mathcal{C}_2 = T^2$, it was already considered in [13]. Note also that the outer-automorphism twist we employ in this paper is around S^1 , not around cycles on \mathcal{C}_2 as was in [14, 15].

structure

$$\mathcal{I} = \sum_{\lambda} \frac{\prod_I K_{\Lambda_I}(\mathbf{a}_I) \underline{P}_{\lambda}(\mathbf{a}_I t^{\Lambda_I})}{K_{\rho} \underline{P}_{\lambda}(t^{\rho})}. \quad (1.1)$$

Here, λ runs over the irreducible representation of the group Γ , \underline{P}_{λ} is the normalized Macdonald polynomial of the corresponding type, Λ_I specifies the type of the I -th puncture, ρ is the type of the null puncture, \mathbf{a}_I is the fugacity of the flavor symmetry of the I -th puncture, $\mathbf{a} t^{\Lambda}$ is determined by the type of the puncture Λ , and $K_{\Lambda}(\mathbf{a})$ is a λ -independent prefactor given by an infinite product

$$K_{\Lambda}(\mathbf{a}) = \prod_{n=0}^{\infty} \prod_s \frac{1}{1 - z_{\Lambda,s}(\mathbf{a}, q, t) q^n}, \quad (1.2)$$

where $z_s(\mathbf{a}, q, t)$ is a monomial of the indicated arguments.

In the twisted case, we analyze cylinders for all types of 6d theory, and three-punctured spheres of type A . Assuming that the twist is of order 2, we will find that the index is given by

$$\tilde{\mathcal{I}} = \sum_{\lambda} \frac{\prod_I \tilde{K}_{\Lambda_I}(\tilde{\mathbf{a}}_I) \tilde{\underline{P}}_{\lambda}(\tilde{\mathbf{a}}_I t^{\Lambda_I})}{\tilde{K}_{\rho} \tilde{\underline{P}}_{\lambda}(t^{\rho})}. \quad (1.3)$$

Here, λ now runs over the irreducible representation of the group G , $\tilde{\underline{P}}_{\lambda}$ is the normalized Macdonald polynomial of the type listed in Table 1, $\tilde{\mathbf{a}}_I$ is the fugacity of the flavor symmetry of the I -th puncture modified so that it is compatible with the twist, and $\tilde{K}_{\Lambda}(\tilde{\mathbf{a}})$ is now given by an infinite product

$$\tilde{K}_{\Lambda}(\tilde{\mathbf{a}}) = \prod_{n=0}^{\infty} \prod_s \frac{1}{1 \pm z_{\Lambda,s}(\tilde{\mathbf{a}}, q, t) q^n}, \quad (1.4)$$

where $z_{\Lambda,s}(\mathbf{a}, q, t)$ is the *same* monomial of the indicated arguments as in (1.2). The choice of a $+$ or a $-$ sign depends on the monomial $z_{\Lambda,s}$; the prescription is given in Sec. 4.2. We will provide ample pieces of evidence supporting the validity of this formula.

The formula is rather complicated but the physical interpretation is straightforward: the outer-automorphism twist σ acts in a natural way on each factor of (1.1). Namely, it modifies the type of the orthogonal polynomial, and it just acts by ± 1 on the Fock space generators contributing to (1.2). This strongly suggests that there is a physical frame where each factor in (1.1) and (1.3) has a simple physical interpretation.

The rest of the paper is organized as follows. In Sec. 2, we review the definition of the superconformal indices of 4d theories, and its TQFT interpretation when the 4d theory considered is of class \mathcal{S} . In Sec. 3, we study the contribution from the vector multiplets, and show how the gauge group G S-dual to the σ -fixed subgroup of Γ appears from a perturbative calculation. In Sec. 4, we study the twisted superconformal indices of free hypermultiplets coming from three-punctured spheres, and check the validity of the formula (1.3). We also perform the check of the agreement of the superconformal index on both sides of the Argyres-Seiberg duality. We have a few appendices detailing the niceties.

2 Twisted superconformal indices and TQFT

In this section, we review the superconformal indices and their 2d TQFT structure and then define the twisted superconformal index.

2.1 Superconformal indices and topological field theory

The superconformal index of $\mathcal{N} = 2$ SCFT is defined as [3, 4]

$$\begin{aligned}\mathcal{I} &= \text{Tr}(-1)^F p^{\frac{1}{2}(E+2j_1-2R-r)} q^{\frac{1}{2}(E-2j_1-2R-r)} t^{R+r} \prod_i x_i^{f_i} \\ &= \text{Tr}(-1)^F \left(\frac{t}{pq}\right)^r p^{j_2+j_1} q^{j_2-j_1} t^R \prod_i x_i^{f_i},\end{aligned}\tag{2.1}$$

where E is the conformal dimension, (j_1, j_2) are the Cartans of the Lorentz rotation $\text{SU}(2)_1 \times \text{SU}(2)_2$, r is the $\text{U}(1)_R$ generator and R is the Cartan generator of $\text{SU}(2)_R$. The f_i are the Cartan generators of the flavor symmetry group. The trace is over the states with $\Delta \equiv E - 2j_2 - 2R + r = 0$ upon radial quantization.

The single letter indices for a vector multiplet and a half-hypermultiplet are given by

$$f^{\text{vect}} = -\frac{p}{1-p} - \frac{q}{1-q} + \frac{pq/t - t}{(1-q)(1-p)},\tag{2.2}$$

$$f^{\frac{1}{2}\text{hyp}} = \frac{\sqrt{t} - pq/\sqrt{t}}{(1-q)(1-p)}.\tag{2.3}$$

For an $\mathcal{N} = 2$ theory with Lagrangian description, one can obtain the superconformal index first by writing a single letter index and then put it in a plethystic exponential

$$\mathcal{I} = \text{PE} \left[\sum_i f_{R_i}(p, q, t) \chi_{R_i}(U) \right]_{p, q, t, U},\tag{2.4}$$

where f_{R_i} are the single letter indices for a $\mathcal{N} = 2$ multiplet in representation R_i of the gauge group. Here, the symbol PE stands for the plethystic exponential, defined by

$$\text{PE} \left[\sum_{n \geq 1} a_n t^n \right]_t = \prod_{n \geq 1} (1 - t^n)^{-a_n}.\tag{2.5}$$

We often omit the subscripts of PE signifying the variable(s) with respect to which the exponential is taken. Subsequently, we also often use the following expression:

$$\text{PE} \left[u \sum_{n \geq 1} a_n t^n \right]_{u=-1} = \prod_{n \geq 1} (1 + t^n)^{-a_n};\tag{2.6}$$

this is also known in the literature as the fermionic plethystic exponential (see *e.g.*, [16]).

The index \mathcal{I} becomes the 2-parameter Macdonald index if we take $p = 0$, and becomes the Schur index when we take $t = q$ as well. Our focus on this paper is the Macdonald index with 2-parameters

$$\mathcal{I}(p, q) = \text{Tr}(-1)^F q^{E-2R-r} t^{R+r} = \text{Tr}(-1)^F q^{-2j_1} t^{R+r}, \quad (2.7)$$

where the trace is over the $\frac{1}{4}$ -BPS states with $\Delta' \equiv E + 2j_1 - 2R - r = 0$ in addition to $\Delta = 0$.

It is straight-forward to evaluate the superconformal index when the Lagrangian description of the theory is available. For example, indices for all the theories in class \mathcal{S} of type A_1 can be evaluated. It is shown in [1] that the indices can be thought of as correlation functions of a 2-dimensional topological field theory on \mathcal{C}_2 used to construct the theory. Moreover, the TQFT structure enables us to evaluate the indices for non-Lagrangian theories as well [2, 3]. The idea of TQFT as applied to this setup is as follows:

1. Any TQFT correlators can be obtained from knowing the three point functions and propagators. A 3-point function can be written as

$$I_{\Lambda_1, \Lambda_2, \Lambda_3}(\mathbf{a}, \mathbf{b}, \mathbf{c}) = \sum_{\lambda\mu\nu} C^{\lambda\mu\nu} f_\lambda(\mathbf{a}; \Lambda_1) f_\mu(\mathbf{b}; \Lambda_2) f_\nu(\mathbf{c}; \Lambda_3), \quad (2.8)$$

where $C^{\lambda\mu\nu}$ is called the structure constant. The propagator is given by

$$\eta(\mathbf{a}, \mathbf{b}) = I^V(\mathbf{a}) \delta(\mathbf{a}, \mathbf{b}^{-1}), \quad (2.9)$$

where $I^V(\mathbf{a})$ is given by the index of a vector multiplet and δ is a Dirac delta function. One can glue the three point functions with propagators to form arbitrary correlators.

2. Choose the basis of functions $\{f^\mu\}$ to be orthogonal under the measure involving the propagator which is the index of a vector multiplet:

$$\eta_{\mu\nu} = \int [d\mathbf{a}][d\mathbf{b}] \eta(\mathbf{a}, \mathbf{b}) f_\mu(\mathbf{a}) f_\nu(\mathbf{b}) = \int [d\mathbf{a}] I^V(\mathbf{a}) f_\mu(\mathbf{a}) f_\nu(\mathbf{a}^{-1}) = \delta_{\mu\nu}, \quad (2.10)$$

where $[d\mathbf{a}]$ is the Haar measure of the gauge group. Once this is done, we can freely move the indices of the structure constant upward or downward.

3. A nontrivial fact is that the structure constant $C^{\lambda\mu\nu}$ can be made diagonal, which means that only $C^{\mu\mu\mu}$ are non-zero. This assures the associativity of TQFT, which in turn guarantees S-dual invariance of the index. Also, this justifies our notation of the structure constant which is independent of the type of punctures on \mathcal{C}_2 .
4. Write $f_\lambda(\mathbf{a}; \Lambda) = K_\Lambda(\mathbf{a}) P_\lambda(\mathbf{a}t^\Lambda)$ where $P_\lambda(\mathbf{a})$ is some known symmetric polynomial orthogonal under certain measure $\Delta'(\mathbf{a})$ and $\mathbf{a}t^\Lambda$ is specified by the type of puncture Λ .

5. There is a type of puncture, ρ , which corresponds to the absence of any puncture. Then the consistency requires that $C^\mu = (K_\rho P_\mu(t^\rho))^{-1}$

Here we omitted the dependence on the fugacities (p, q, t) from the notation.

Since we know how the orthogonal polynomial $P_\lambda(\mathbf{a})$ generalizes to higher rank, we can write down a conjecture for the superconformal indices of the theories in class \mathcal{S} . The non-trivial piece of information we need to determine is then the prescription for $K_\Lambda(\mathbf{a})$ and $\mathbf{a}t^\Lambda$ for generic type of punctures.

Once they are given, we can write the superconformal index of a theory in class \mathcal{S} defined by a Riemann surface \mathcal{C}_2 of genus g and s punctures of type $\Lambda_{1,\dots,s}$ as

$$\mathcal{I} = \sum_{\lambda} \frac{\prod_{I=1}^s K_{\Lambda_I}(\mathbf{a}_I) P_{\lambda}(\mathbf{a}_I t^{\Lambda_I})}{(K_{\rho} P_{\lambda}(t^{\rho}))^{2g-2+s}}. \quad (2.11)$$

In the following, we will define the superconformal index with outer-automorphism twist and then write it as a correlation function of a 2d topological field theory as the untwisted case.

2.2 Superconformal indices with outer-automorphism twist

We define the twisted superconformal index as

$$I = \text{Tr}(-1)^F \mathcal{P}_{\sigma} q^m t^n \prod_i x_i^{f_i}. \quad (2.12)$$

where \mathcal{P}_{σ} acts as outer-automorphism σ on the gauge group Γ and $m = -2j_1, n = R + r$. Upon twisting, the index should get contributions only from the eigenstates of \mathcal{P}_{σ} .

Let Γ be the gauge group of the theory, and G be the group obtained by folding the Dynkin diagram of Γ or the S-dual of σ -invariant subgroup. We assume Γ to be simply-laced. Let's denote the number of simple roots of Γ invariant under σ to be r' and the rank of Γ to be r . Write $r = r' + dr''$, where d is the order of the outer-automorphism group of Γ . Then the rank of G is given by $r' + r''$. For $\Gamma = A_{\ell}, D_{\ell}, E_6$ the outer-automorphism group is given by \mathbb{Z}_2 and for D_4 , one can have \mathbb{Z}_3 .

Suppose we label non-gauge invariant states contributing to the Macdonald index as $v_{\alpha;m,n}$ where m, n are the eigenvalues of the corresponding operators, and α is a weight of the gauge group. At the end of day, we should pick up gauge invariant states only. We will discuss it later in this section. For the time being, we regard the gauge symmetry as a flavor symmetry and introduce the fugacities x associated to it to the index. We will use the multiplicative notation for the fugacities. So, for a vector α_i and fugacities x_i , x^{α} stands for the product $\prod_i x_i^{\alpha_i}$.

The outer-automorphism σ becomes an operator \mathcal{P}_{σ} acting on the Fock space. Here we consider its actions on the modes which transform as adjoints, i.e. $\alpha \in \Delta$, where Δ is the set of roots. Other representations can be considered similarly, and we will encounter them in later sections. If the root α is sent to β by σ , the mode $v_{\alpha;m,n}$ is sent to $v_{\beta;m,n}$. The effect

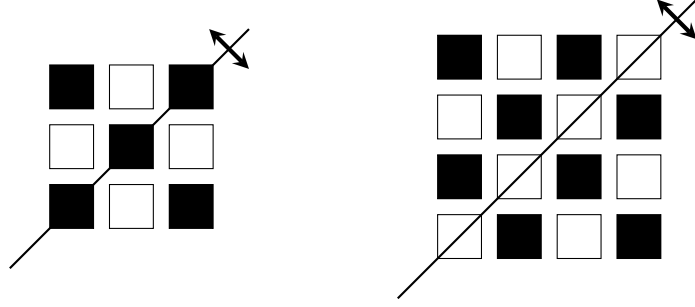


Figure 1: The \mathbb{Z}_2 action (2.15) and (2.17) on the matrix. The sign is plus or minus if the entry is white or black, respectively.

of \mathcal{P}_σ to the index for the case of $\Gamma \neq A_{2\ell}$ and $\Gamma = A_{2\ell}$ should be treated separately. This distinction arises because when $\beta = \sigma(\alpha) \neq \alpha$, β and α are orthogonal, *unless* $\Gamma = A_{2\ell}$.

For $\Gamma \neq A_{2\ell}$: Let Γ be one of $A_{2\ell-1}$, D_{n+1} , D_4 or E_6 . Then the trace with the insertion of \mathcal{P}_σ is given by the product of the following factors: (Note that x is now restricted to be invariant under σ .)

- If α is invariant under σ , we just have the same contribution $x^\alpha q^m t^n$.
- If σ is \mathbb{Z}_2 and exchanges α and β , we need to study the trace of \mathcal{P}_σ acting on the states

$$(v_{\alpha;m,n})^k (v_{\beta;m,n})^l. \quad (2.13)$$

The trace only receives the contribution from the states with $k = l$, so it's a sum over

$$(v_{\alpha;m,n} v_{\beta;m,n})^n. \quad (2.14)$$

For each such state we get the contribution $x^{\alpha+\beta} q^{2m} t^{2n}$.

- If σ is \mathbb{Z}_3 and permutes $\alpha \rightarrow \beta \rightarrow \gamma \rightarrow \alpha$, similarly it gives $x^{\alpha+\beta+\gamma} q^{3m} t^{3n}$.

Here we record the explicit action of σ on $A_{2\ell-1}$, as we will need it later. Let us call the entries of 2ℓ -by- 2ℓ matrix e_{ij} , where $i, j = -\ell, -(\ell-1), \dots, -1, 1, 2, \dots, \ell-1, \ell$. Then we pick σ to act by

$$e_{i,j} \rightarrow -(-1)^{s(i)-s(j)} e_{-j,-i} \quad (2.15)$$

where

$$s(i) = \begin{cases} i, & (i > 0) \\ i+1. & (i < 0) \end{cases} \quad (2.16)$$

The invariant subalgebra is $\text{USp}(2\ell)$, i.e C_ℓ . Its S-dual is B_ℓ . Note that the action (2.15) flips the matrix along its anti-diagonal, and the sign is given by the checker board pattern so that it is -1 on the diagonal, see Fig. 1.

For $\Gamma = A_{2\ell}$: this case needs to be treated separately. Let us call the entries of $(2\ell + 1)$ -by- $(2\ell + 1)$ matrix e_{ij} , where $i, j = -\ell, -(\ell - 1), \dots, -1, 0, 1, 2, \dots, \ell - 1, \ell$. Then one particularly nice outer automorphism σ is given by

$$e_{i,j} \rightarrow -(-1)^{i-j} e_{-j,-i}. \quad (2.17)$$

The invariant subalgebra is $\text{SO}(2\ell + 1)$, i.e. B_ℓ . Its S-dual is C_ℓ . See Fig. 1. We let the Cartan subalgebra to be formed by $e_{i,i}$. Then the invariant part is given by

$$a_i = e_{i,i} = -e_{-i,-i} \quad (2.18)$$

for $i = 1, \dots, n$. $e_{0,0}$ is forced to be zero.

Let's denote by $v_{i,j}$ the modes corresponding to $e_{i,j}$, where $i \neq j$. We suppress the dependence on m, n for simplicity. The mode $v_{i,j}$ has charge $a_i - a_j$, where we define $a_{-i} = -a_i$. Then \mathcal{P}_σ acts on $v_{i,j}$ by

$$v_{i,-i} \rightarrow -v_{i,-i} \quad (2.19)$$

for $i = -n, \dots, n$ (excluding $i = 0$) and by

$$v_{i,j} \rightarrow -v_{-j,-i} \quad (2.20)$$

for $i \neq -j$. Then the combinations of v 's contributing to the trace are

	charge	action of \mathcal{P}_σ	index
$v_{i,-i}$	$2a_i$	$-$	$-x_i^2 q^m t^n$
$v_{i,0} v_{0,-i}$	$2a_i$	$+$	$x_i^2 q^{2m} t^{2n}$
$v_{-i,i}$	$-2a_i$	$-$	$-x_i^{-2} q^m t^n$
$v_{-i,0} v_{0,i}$	$-2a_i$	$+$	$x_i^{-2} q^{2m} t^{2n}$
$v_{i,j} v_{-j,-i}$	$2(a_i - a_j)$	$+$	$x_i^2 x_j^{-2} q^{2m} t^{2n}$
$v_{i,-j} v_{j,-i}$	$2(a_i + a_j)$	$+$	$x_i^2 x_j^2 q^{2m} t^{2n}$
$v_{-i,j} v_{-j,i}$	$2(-a_i - a_j)$	$+$	$x_i^{-2} x_j^{-2} q^{2m} t^{2n}$

(2.21)

where i and j are both positive.

2.2.1 Haar measures

In order to pick up the gauge invariant states, we need to integrate over the index obtained from non-gauge invariant states with the appropriate Haar measure. Let us recall how the Haar measure arises in the calculation of the superconformal index:

$$\int [d\mathbf{x}]_\Gamma = \frac{1}{\#W} \int \prod_{a=1}^r \frac{dx_a}{2\pi i x_a} \prod_{\alpha \in \Delta_\Gamma} (1 - x^\alpha) \quad (2.22)$$

where the product is over the roots α of Γ and $\#W$ is the order of the Weyl group of type Γ and the integration is over the unit circle in a complex plane. The point is that the

Wilson line x_a around S^1 of the Γ gauge field, gauge-fixed to lie in the Cartan subgroup, is a zero mode to be integrated over. Then the off-diagonal elements of vector multiplets of Γ , independent along S^3 , give one-loop fluctuations, by creating a Fock space of Fadeev-Popov ghosts. Say a root α gives a fermionic raising operator b_α . Then we have a pair of states

$$|\psi\rangle, \quad b_\alpha|\psi\rangle, \quad (2.23)$$

for any state $|\psi\rangle$. The Wilson line works as a chemical potential, and a mode b_α contributes to the partition function by x^α . Then the contribution from v_α to the partition function is $(1 - x^\alpha)$, accounting for a factor in (2.22).

Let us now study how the Haar measure of the Γ gauge group changes if we include the twist by σ . We will discuss the case of $\Gamma \neq A_{2\ell}$ and $\Gamma = A_{2\ell}$ separately as we have discussed.

For $\Gamma \neq A_{2\ell}$: Now, the Haar measure (2.22) becomes

$$\int d\mathbf{x} \prod_{\alpha=\sigma(\alpha)} (1 - x^\alpha) \prod_{\alpha \leftrightarrow \beta} (1 - x^{\alpha+\beta}) \quad (2.24)$$

when σ is order 2, or

$$\int d\mathbf{x} \prod_{\alpha=\sigma(\alpha)} (1 - x^\alpha) \prod_{\alpha \rightarrow \beta \rightarrow \gamma} (1 - x^{\alpha+\beta+\gamma}) \quad (2.25)$$

when σ is order 3. In both cases, the integral is over the σ -invariant subspace of the Cartan subgroup of Γ . As is well known and is explained in the Appendix A of [17], the expressions above become

$$\int [d\mathbf{x}]_G = \int d\mathbf{x} \prod_{\alpha \in \Delta_G} (1 - x^\alpha) \quad (2.26)$$

where α runs over the roots of G , and the integral of x is over the Cartan of G . Thus we see that, in some sense, the outer-automorphism σ binds the mode v_α and v_β to a different mode $v_\alpha v_\beta$, to produce the charge of the gauge group G which is S-dual to the subgroup invariant under σ . This appearance of the Haar measure for G can be studied from another point of view, see Appendix A.

For $\Gamma = A_{2\ell}$: The first two modes of (2.21) give the contribution

$$(1 - x_i^2)(1 + x_i^2) = (1 - x_i^4), \quad (2.27)$$

the second two modes

$$(1 - x_i^{-2})(1 + x_i^{-2}) = (1 - x_i^{-4}), \quad (2.28)$$

and the last three modes give

$$\prod_{i>j,\pm,\pm} (1 - x_i^{\pm 2} x_j^{\pm 2}). \quad (2.29)$$

The total contribution is then

$$\prod_{\alpha} (1 - (x^2)^{\alpha}) \quad (2.30)$$

where

$$\alpha = \pm 2a_i, \quad \pm a_i \pm a_j \quad (2.31)$$

which is the measure factor for C_ℓ , as it should be. Note that the contribution for the long root $2a_i$ of C_ℓ comes from the combination of the modes $e_{i,-i}$ and $e_{i,0}e_{0,-i}$. Note also that the fugacity x^2 is rescaled by a factor of two, whereas when the C_ℓ Haar measure arises from $\Gamma = D_{n+1}$, there is no such rescaling, see (2.26).

3 Contributions from vector multiplets

3.1 Without outer-automorphism twist

Let us first study the contribution from the vector multiplets, or equivalently the cylinder in the Riemann surface. A vector multiplet of the gauge group G gives

$$\int [\mathrm{d}\mathbf{x}]_G \mathrm{PE} \left[\frac{-p - q + 2pq + pq/t - t}{(1-p)(1-q)} \chi_{\mathrm{adj}}(\mathbf{x}) \right]_{p,q,t,x} \quad (3.1)$$

as can be seen from (2.3) and (2.4). Here, $[\mathrm{d}\mathbf{x}]_G$ is the Haar measure for the group G as in the last section.

The two-parameter version of the measure is obtained by setting $p = 0$:

$$\int [\mathrm{d}\mathbf{x}] \mathrm{PE} \left[\frac{-q - t}{1 - q} \chi_{\mathrm{adj}}(\mathbf{x}) \right] \quad (3.2)$$

and the one-parameter version is obtained by further setting $q = t$:

$$\int [\mathrm{d}\mathbf{x}] \mathrm{PE} \left[\frac{-2q}{1 - q} \chi_{\mathrm{adj}}(\mathbf{x}) \right]. \quad (3.3)$$

In the superconformal index, it is important to obtain a orthogonal measure under (3.1), (3.2), or (3.3). But note that (3.3) is just

$$\int [\mathrm{d}\mathbf{x}] \mathrm{PE} \left[\frac{-q}{1 - q} \chi_{\mathrm{adj}}(\mathbf{x}) \right]^2. \quad (3.4)$$

Therefore, the orthogonal polynomials under it are just

$$\mathrm{PE} \left[\frac{+q}{1 - q} \chi_{\mathrm{adj}}(\mathbf{x}) \right] \chi_{\lambda}(\mathbf{x}), \quad (3.5)$$

where $\chi_{\lambda}(\mathbf{x})$ is the character of an irreducible representation (irrep) of highest weight λ .

For the two-parameter case (3.2), a way out is to use Macdonald polynomials P_λ of type G , which are defined as the orthogonal polynomials under the measure

$$\int [d\mathbf{x}] \prod_{n=0}^{\infty} \prod_{\alpha} \frac{1 - q^{n+1} x^\alpha}{1 - tq^n x^\alpha}. \quad (3.6)$$

Now rewrite (3.2) as (where r is the rank of G)

$$(3.2) = \text{PE} \left[\frac{-q+t}{1-q} \right]^r \int [d\mathbf{x}] \text{PE} \left[\frac{-q+t}{1-q} \sum_{\alpha \in \Delta} x^\alpha \right] \text{PE} \left[\frac{-t}{1-q} \chi_{\text{adj}}(\mathbf{x}) \right]^2. \quad (3.7)$$

This makes it clear that dressed Macdonald polynomials

$$K_{\text{full}}(\mathbf{x}) P_\lambda(\mathbf{x}) \quad (3.8)$$

where

$$K_{\text{full}}(\mathbf{x}) = \text{PE} \left[\frac{t}{1-q} \chi_{\text{adj}}(\mathbf{x}) \right] \quad (3.9)$$

are orthogonal under the two-parameter vector-multiplet measure (3.2). The factor $K_{\text{full}}(\mathbf{x})$ will recur as the prefactor for the full puncture later.

3.2 With outer-automorphism twist

3.2.1 Macdonald measures

The analysis of the whole vector multiplet measure goes almost verbatim as that of the Haar measure. But before doing so, we need to recall basic properties of Macdonald measures and polynomials of general type [18]. These polynomials are *not* labeled by finite Dynkin diagram; rather, they correspond to (possibly non-reduced) affine root systems S . A reduced affine root system is the usual affine root system underlying (un)twisted affine Lie algebras as discussed by Kac [19]. A non-reduced root system is a root system where α and 2α can simultaneously be roots; a non-reduced affine root systems S is labeled by a pair of reduced affine root systems (X, Y) where $X = \{\alpha \in S : \alpha/2 \notin S\}$, $Y = \{\alpha \in S : 2\alpha \notin S\}$. These are tabulated in Table 2.

Reduced:

Kac	$X_\ell^{(1)}$	$A_{2\ell-1}^{(2)}$	$A_{2\ell}^{(2)}$	$D_{n+1}^{(2)}$	$E_6^{(2)}$	$D_4^{(3)}$
Macdonald	X_ℓ	B_ℓ^\vee	BC_ℓ	C_ℓ^\vee	F_4^\vee	G_2^\vee

Non-reduced:

$$(BC_\ell, C_\ell), \quad (C_\ell^\vee, BC_\ell), \quad (B_\ell, B_\ell^\vee), \quad (C_\ell^\vee, C_\ell).$$

Table 2: List of (possibly-non-reduced) affine root systems

The measure for the Macdonald polynomials of various types is given in [18], (5.1.28). For simply-laced X_ℓ , it was given in (3.6). For non-simply-laced $X = G$, it is

$$\int [\mathbf{d}\mathbf{x}]_{G,q,t,t'} = \int [\mathbf{d}\mathbf{x}]_G \prod_{n=0}^{\infty} \prod_{\alpha \in \Delta_s} \frac{1 - q^{n+1}x^\alpha}{1 - tq^n x^\alpha} \prod_{\alpha \in \Delta_l} \frac{1 - q^{(n+1)}x^\alpha}{1 - t'q^n x^\alpha} \quad (3.10)$$

where t, t' for the short roots and the long roots can be different.

The measure for type G^\vee is given by

$$\int [\mathbf{d}\mathbf{x}]_{G^\vee,q,t,t'} = \int [\mathbf{d}\mathbf{x}]_{G^\vee} \prod_{n=0}^{\infty} \prod_{\alpha \in \Delta_s} \frac{1 - q^{n+1}x^\alpha}{1 - tq^n x^\alpha} \prod_{\alpha \in \Delta_l} \frac{1 - q^{(n+1)d}x^\alpha}{1 - t'q^{nd}x^\alpha} \quad (3.11)$$

where $\Delta_{s,l}$ are short and long roots for G^\vee (as a finite root system). Note that we have q^d for the long roots instead of q , where $d = 2$ for $G^\vee = B, C, F$ and $d = 3$ for $G^\vee = G_2$.

The Macdonald polynomials associated to any non-reduced affine root system can be obtained by restricting the parameters of the Macdonald polynomials of type (C_ℓ^\vee, C_ℓ) , which are also called the Koornwinder polynomials. It has the measure

$$\int [\mathbf{d}\mathbf{x}]_{\text{Koornwinder}} = \int [\mathbf{d}\mathbf{x}]_{C_\ell} \text{PE} \left[\sum_{\alpha \in \Delta_s} \frac{\check{t} - \check{q}}{1 - \check{q}} x^\alpha + \sum_{\alpha \in \Delta_l} \left(\frac{u_1 + u_2 + u_3 + u_4}{1 - \check{q}} x^{\alpha/2} + \frac{-\check{q}}{1 - \check{q}} x^\alpha \right) \right] \quad (3.12)$$

where $(\check{q}, \check{t}, u_1, u_2, u_3, u_4)$ are parameters.

3.2.2 Vector multiplet measures

In this section, we show that the orthogonal polynomial under the twisted vector multiplet measure is given by

$$\tilde{K}_{\text{full}}(\mathbf{x}) \tilde{P}_\lambda(\mathbf{x}), \quad (3.13)$$

where $\tilde{K}_{\text{full}}(\mathbf{x})$ is a prefactor given in (3.16), (3.23) and $\tilde{P}_\lambda(\mathbf{x})$ is the Macdonald polynomials of type listed in Table 3. As always, the discussion is split into two cases, depending on whether $\Gamma = A_{2\ell}$.

For $\Gamma \neq A_{2\ell}$: Consider the two-parameter vector multiplet measure for simply-laced $\Gamma \neq A_{2\ell}$ and the \mathbb{Z}_d twist σ , as in Sec. 2.2.1. Using r', r'' such that $r = r' + r''$ is the rank of G and $r' + dr''$ is the rank of Γ , we obtain

$$\text{PE} \left[\frac{-q - t}{1 - q} \right]^{r'} \text{PE} \left[\frac{-q^d - t^d}{1 - q^d} \right]^{r''} \int [\mathbf{d}\mathbf{x}]_G \prod_{\alpha \in \Delta_s} \text{PE} \left[\frac{-q - t}{1 - q} x^\alpha \right] \prod_{\alpha \in \Delta_l} \text{PE} \left[\frac{-q^d - t^d}{1 - q^d} x^\alpha \right] \quad (3.14)$$

which is

$$\text{PE} \left[\frac{-q + t}{1 - q} \right]^{r'} \text{PE} \left[\frac{-q^d + t^d}{1 - q^d} \right]^{r''} \int [\mathbf{d}\mathbf{x}]_{S,q,t,t'=t^d} \tilde{K}_{\text{full}}(\mathbf{x})^{-2} \quad (3.15)$$

Γ	$A_{2\ell-1}$	D_{n+1}	D_4	E_6	$A_{2\ell}$
d	2	2	3	2	2
S	C_ℓ^\vee	B_ℓ^\vee	G_2^\vee	F_4^\vee	(C_ℓ^\vee, C_ℓ)
G	B_ℓ	C_ℓ	G_2	F_4	C_ℓ

Table 3: Types of the Macdonald polynomials appearing in the vector multiplet measure. Upon 1-parameter specialization $t = q$, we get the Schur functions of type G .

where

$$\tilde{K}_{\text{full}}(\mathbf{x}) = \text{PE} \left[\frac{t}{1-q} \right]^{r'} \text{PE} \left[\frac{t^d}{1-q^d} \right]^{r''} \prod_{\alpha \in \Delta_s} \text{PE} \left[\frac{t}{1-q} x^\alpha \right] \prod_{\alpha \in \Delta_l} \text{PE} \left[\frac{t^d}{1-q^d} x^\alpha \right]. \quad (3.16)$$

For $\Gamma \neq A_{2\ell}$: The vector multiplet measure in this case is

$$\begin{aligned} & \text{PE} \left[\frac{-q-t}{1-q} \right]^{r'} \text{PE} \left[\frac{-q^d-t^d}{1-q^d} \right]^{r''} \int [\mathbf{d}\mathbf{x}]_{C_\ell} \times \\ & \text{PE} \left[\sum_{\alpha \in \Delta_s} \frac{-q^2-t^2}{1-q^2} x^\alpha + \sum_{\alpha \in \Delta_l} \left(\frac{-q^2-t^2}{1-q^2} x^{\alpha/2} + \frac{-uq-ut}{1-q} x^{\alpha/2} \right) \right]_{u=-1}. \end{aligned} \quad (3.17)$$

The first factor in the second line can be rewritten as

$$\text{PE} \left[\frac{-q^2-t^2}{1-q^2} x^\alpha \right] = \text{PE} \left[\frac{-\check{q}+\check{t}}{1-\check{q}} x^\alpha \right] \text{PE} \left[\frac{-\check{t}}{1-\check{q}} x^\alpha \right]^2 \quad (3.18)$$

and the second factor can be massaged as

$$\begin{aligned} & \text{PE} \left[\left(\frac{-q^2-t^2}{1-q^2} + \frac{-uq-ut}{1-q} \right) x^{\alpha/2} \right]_{u=-1} \\ & = \text{PE} \left[\left(\frac{-q^2+t^2}{1-q^2} + \frac{-uq+ut}{1-q} \right) x^{\alpha/2} \right]_{u=-1} \text{PE} \left[\left(\frac{-t^2}{1-q^2} + \frac{-ut}{1-q} \right) x^{\alpha/2} \right]_{u=-1}^2. \end{aligned} \quad (3.19)$$

This last expression can be further rewritten as

$$\begin{aligned} & \text{PE} \left[\left(\frac{-q^2+t^2}{1-q^2} + \frac{-uq+ut}{1-q} \right) x^{\alpha/2} \right]_{u=-1} \\ & = \text{PE} \left[\frac{u_1+u_2+u_3+u_4}{1-\check{q}} x^{\alpha/2} + \frac{-\check{q}}{1-\check{q}} x^\alpha \right]_{(\check{q}, \check{t}; u_1, u_2, u_3, u_4) = (q^2, t^2; -tq, -t, q, t^2)}. \end{aligned} \quad (3.20)$$

Combining all this, we find that the twisted vector multiplet measure equals

$$\text{PE} \left[\frac{-q+t}{1-q} \right]^{r'} \text{PE} \left[\frac{-q^d+t^d}{1-q^d} \right]^{r''} \int [\mathbf{d}\mathbf{x}]_{\text{Koornwinder}} \tilde{K}_{\text{full}}(\mathbf{x})^{-2} \quad (3.21)$$

where

$$[d\mathbf{x}]_{\text{Koornwinder}} = [d\mathbf{x}]_{(C_\ell^\vee, C_\ell), (\tilde{q}, \tilde{t}; u_1, u_2, u_3, u_4) = (q^2, t^2; -tq, -t, q, t^2)} \quad (3.22)$$

is the Koornwinder measure (3.12), with the indicated choice of the parameters, and

$$\tilde{K}_{\text{full}}(\mathbf{x}) = \text{PE} \left[\frac{t^2}{1-q^2} \right]^{r''} \prod_{\alpha \in \Delta_s} \text{PE} \left[\frac{t}{1-q} x^\alpha \right] \prod_{\alpha \in \Delta_l} \text{PE} \left[\frac{ut}{1-q} x^{\alpha/2} + \frac{t^2}{1-q^2} x^{\alpha/2} \right]_{u=-1}. \quad (3.23)$$

Let us make some observations. First, the vector multiplet measure for $G = C_\ell$ obtained from $\Gamma = A_{2\ell}$ is different from the one for $G = C_\ell$ obtained from $\Gamma = D_{n+1}$. From the point of view of the 5d maximally-supersymmetric Yang-Mills, this should stem from the difference in the discrete theta angle taking value in $\pi_4(G) = \mathbb{Z}_2$. Second, the measure in the one-parameter version can easily be obtained by setting $t = q$ in the formulas above. The vector multiplet measures for the two ways of obtaining $G = C_\ell$ still differ.

4 Contributions from the matter fields

Let us now move on to the study of the 4d theory corresponding to a general Riemann surface \mathcal{C}_2 with punctures. Any Riemann surface can be decomposed to cylinders and three-punctured spheres, and we already discussed the contribution from the cylinder in the last section. Therefore, we have to study the superconformal indices of the 4d theories corresponding to three-punctured spheres, which we call as matter fields.

4.1 Without outer-automorphism twists

A puncture for the 6d theory of type $G = A_{n-1}$ is specified by a Young diagram Λ with n boxes, or equivalently an embedding

$$\Lambda : \text{SU}(2) \rightarrow G. \quad (4.1)$$

The partition $\mathbf{N} = [l_1, l_2, \dots]$ means that under the corresponding $\Lambda : \text{SU}(2) \rightarrow \text{SU}(N)$, the fundamental of $\text{SU}(N)$ decomposes into a direct sum of irreducible representations of $\text{SU}(2)$ of dimension l_1, l_2, \dots respectively.

The flavor symmetry G_Λ associated to the puncture is given by the commutant of $\Lambda(\text{SU}(2))$ in G :

$$G_\Lambda \times \Lambda(\text{SU}(2)) \subset G. \quad (4.2)$$

Thus, the fugacity a of the flavor symmetry of a puncture can naturally be thought of as a fugacity of $\text{SU}(n)$. The superconformal index of the theory for the three-punctured sphere with punctures $\Lambda_{1,2,3}$ is conjectured to have the following form [3]:

$$\mathcal{I} = \sum_{\lambda} \frac{\prod_{I=1}^3 K_{\Lambda_I}(\mathbf{a}_I) \underline{P}_{\lambda}(\mathbf{a}_I t^{\Lambda_I})}{K_{\rho} \underline{P}_{\lambda}(t^{\rho})}. \quad (4.3)$$

Here, we made the obvious generalization of the formula in [3] to a general simply-laced group G , and our $K_{\mathbf{A}}(\mathbf{a})$ is different from their $\hat{K}_{\mathbf{A}}(\mathbf{a})$ by an a -independent (q, t) -dependent factor. Let us explain the notations.

- $\underline{P}_{\lambda}(\mathbf{x}) = N_{\lambda}^{-1/2} P_{\lambda}(\mathbf{x})$ is the normalized Macdonald polynomial orthonormal under the measure

$$\frac{(q; q)^r}{(t; q)^r} [\mathrm{d}\mathbf{x}]_{G,t,q} = [\mathrm{d}\mathbf{x}]_G \mathrm{PE} \left[\frac{-q+t}{1-q} \chi_{\mathrm{adj}}(\mathbf{x}) \right] \quad (4.4)$$

where

- $(a; q)$ is the q -Pochhammer symbol

$$(a; q) = \prod_{i=0}^{\infty} (1 - aq^i). \quad (4.5)$$

- r is the rank of G .
- We follow the convention in the mathematics literature by reserving the letter $P_{\lambda}(\mathbf{x})$ to be the unnormalized orthogonal Macdonald polynomial satisfying

$$P_{\lambda}(\mathbf{x}) = \chi_{\lambda}(\mathbf{x}) + \sum_{\mu < \lambda} f_{\lambda}(q, t) \chi_{\mu}(\mathbf{x}), \quad (4.6)$$

where the sum is over μ such that $\lambda - \mu$ is a sum of positive number of simple roots. The series expansions of $P_{\lambda}(\mathbf{x})$ and N_{λ} in terms of q and t are known to have integer coefficients. This guarantees that the series expansion of the formula (4.3) has integer coefficients, as it should be for a superconformal index.

- The argument of the Macdonald polynomials in the numerator is given by $\mathbf{a}t^{\mathbf{A}}$. Here, $\mathbf{a} \in G_{\mathbf{A}}$ and t is now thought of as the element $t \in \mathrm{SU}(2)$ given by $\mathrm{diag}(t^{1/2}, t^{-1/2})$. Then $t^{\mathbf{A}}$ is defined to be the image of $t \in \mathrm{SU}(2)$ via the map (4.1) in $\mathbf{A}(\mathrm{SU}(2))$. Then $\mathbf{a}t^{\mathbf{A}}$ is naturally an element of G via the embedding (4.2).

The physical reason behind the factor $t^{\mathbf{A}}$ is as follows. The puncture of type \mathbf{A} , as a boundary condition of 5d maximally supersymmetric Yang-Mills of gauge group G , is described by giving the profile [20, 21]

$$\Phi_i(s) = \mathbf{A}(\tau_i)/s, \quad (4.7)$$

where τ_i are the standard generators of $\mathrm{SU}(2)$, s is the distance to the boundary, and $\Phi_{1,2,3}$ are three scalars of the theory, transforming as a triplet under the $\mathrm{SU}(2)_R \subset \mathrm{SO}(5)_R$ symmetry of the 5d Yang-Mills. This profile is only invariant under the diagonal subgroup $\mathrm{SU}(2)_R \times \mathbf{A}(\mathrm{SU}(2))$, which becomes the $\mathrm{SU}(2)_R$ symmetry of the resulting 4d theory. From the definition of the $\mathcal{N} = 2$ superconformal index in (2.1), it is then clear that t also appears as a fugacity of the G -flavor symmetry.

- In the denominator, t^ρ is defined using the principal embedding

$$\rho : \mathrm{SU}(2) \rightarrow G \quad (4.8)$$

and in particular

$$t^\rho = \mathrm{diag}(t^{(n-1)/2}, t^{(n-3)/2}, \dots, t^{(1-n)/2}) \quad (4.9)$$

for $G = \mathrm{SU}(n)$. The exponents are given by the coordinates of the Weyl vector.

- $K_\Lambda(\mathbf{a})$ is a prefactor independent of λ , determined as follows. The embedding (4.1), (4.2) induces the decomposition

$$\mathfrak{g} = \bigoplus_j R_j \otimes V_j, \quad (4.10)$$

where \mathfrak{g} is the Lie algebra of G , R_j is a representation of G_Λ and V_{2j+1} is the spin- j irreducible representation of $\mathrm{SU}(2)$. Note that $\bigoplus_j R_j$ gives the decomposition of the Slodowy slice [21]. Each component in the slice give rise to a plethystic exponential, giving

$$K_\Lambda(\mathbf{a}) = \mathrm{PE} \left[\sum_j \frac{t^{j+1}}{1-q} \mathrm{tr}_{R_j}(\mathbf{a}) \right]. \quad (4.11)$$

- For the full puncture, the corresponding embedding $\Lambda_{\mathrm{full}} : \mathrm{SU}(2) \rightarrow G$ is the zero map. Then

$$K_{\mathrm{full}}(\mathbf{a}) = \mathrm{PE} \left[\frac{t}{1-q} \chi_{\mathrm{adj}}(\mathbf{a}) \right]. \quad (4.12)$$

Note that $R_\lambda(\mathbf{a}) \equiv K_{\mathrm{full}}(\mathbf{a}) \underline{P}_\lambda(\mathbf{a})$ satisfies the orthonormality condition under the vector multiplet measure (3.2) thanks to the equality (3.7):

$$\int [\mathrm{d}\mathbf{x}]_G \mathrm{PE} \left[\frac{-q-t}{1-q} \chi_{\mathrm{adj}}(\mathbf{x}) \right] R_\lambda(\mathbf{x}) R_\mu(x^{-1}) = \delta_{\lambda\mu}. \quad (4.13)$$

- For the zero puncture, the corresponding embedding is given by $\rho : \mathrm{SU}(2) \rightarrow G$ is the principal embedding. Then the decomposition (4.10) is

$$\mathfrak{g} = \bigoplus_{i=1}^r V_{d_i-1}, \quad (4.14)$$

where the integers $d_{1,\dots,r}$ are the degrees of invariants of G , so that $d_i = i + 1$ when $G = A_\ell$. Then

$$K_\rho = \mathrm{PE} \left[\sum_i \frac{t^{d_i}}{1-q} \right] = \prod_{i=1}^r (t^{d_i}; q)^{-1}. \quad (4.15)$$

This also appears in the denominator.

From the orthonormality (4.13), the superconformal index for the Riemann surface with genus g and s punctures $\Lambda_{I=1,\dots,s}$ can easily be computed:

$$\mathcal{I} = \sum_{\lambda} \frac{\prod_{I=1}^s K_{\Lambda_I}(\mathbf{a}_I) \underline{P}_{\lambda}(\mathbf{a}_I t^{\Lambda_I})}{[K_{\rho} \underline{P}_{\lambda}(t^{\rho})]^{2(g-1)+s}}. \quad (4.16)$$

This expression also makes it clear that adding a puncture of type ρ , which corresponds to the absence of any puncture, makes no change in the partition function.

4.2 With outer-automorphism twists

Let us generalize the results (4.3) and (4.16) in the previous section to the type- A theories with twists. Here we will first state the final formula. Ample pieces of evidence are given later, by considering various three-punctured spheres giving free hypermultiplets, and the Argyres-Seiberg duality.

Take the 4d theory arising from a sphere of the A_{N-1} theory with three punctures of type $\Lambda_{1,2,3}$ respectively. We find that the superconformal index with the \mathbb{Z}_2 twist is given by

$$\tilde{\mathcal{I}} = \sum_{\lambda} \frac{\prod_{I=1}^3 \tilde{K}_{\Lambda_I}(\mathbf{a}_I) \tilde{\underline{P}}_{\lambda}(\mathbf{a}_I t^{\Lambda_I})}{\tilde{K}_{\rho} \tilde{\underline{P}}_{\lambda}(t^{\rho})}. \quad (4.17)$$

The symbols are explained below:

- $\tilde{\underline{P}}_{\lambda} = \tilde{N}_{\lambda}^{-1/2} \tilde{P}_{\lambda}$ are the normalized Macdonald polynomials, orthonormal under the measure

$$\frac{(q; q)^{r'}}{(t; q)^{r'}} \frac{(q^2; q^2)^{r''}}{(t^2; q^2)^{r''}} [\mathrm{d}\mathbf{x}]_{\text{Macdonald}}. \quad (4.18)$$

Here, $[\mathrm{d}\mathbf{x}]_{\text{Macdonald}}$ is the Macdonald measure of the type and the parameters as defined in Sec. 3.2.2. Also, as given in that section, r' and $2r''$ are the dimensions of the \mathbb{Z}_2 invariant and non-invariant parts of the Cartan. Concretely, we have

$$(r', r'') = (1, l-1) \quad \text{if } N = 2l, \quad (r', r'') = (0, l) \quad \text{if } N = 2l+1. \quad (4.19)$$

Here \tilde{P}_{λ} are the unnormalized Macdonald polynomials. When $t = q$, they reduce to characters of B_l for $N = 2l$ and of C_l for $N = 2l+1$. The way to calculate them is described in Appendix B.

We use the convention that

$$\begin{aligned} \chi_{\text{fund}}^{C_l}(\mathbf{z}) &= E_1(\mathbf{z}), & \chi_{\text{vec}}^{B_l}(\mathbf{z}) &= 1 + E_1(\mathbf{z}), \\ \chi_{\text{adj}}^{B_l}(\mathbf{z}) &= l + E_1(\mathbf{z}) + E_2(\mathbf{z}), & \chi_{\text{spin}}^{B_l}(\mathbf{z}) &= \hat{E}_l(\mathbf{z}). \end{aligned}$$

Here and in the following, we use the abbreviations

$$E_1(\mathbf{z}) = \sum_{i=1}^l z_i^2 + z_i^{-2},$$

$$\begin{aligned}
E_2(\mathbf{z}) &= \sum_{1 \leq i < j \leq l} z_i^2 z_j^2 + z_i^2 z_j^{-2} + z_i^{-2} z_j^2 + z_i^{-2} z_j^{-2} , \\
\widehat{E}_l(\mathbf{z}) &= \sum_{\epsilon_i = \pm 1} z_1^{\epsilon_1} \dots z_l^{\epsilon_l} .
\end{aligned} \tag{4.20}$$

- In the untwisted case we had $P_\lambda(\mathbf{a})$ in both the denominator and the numerator, where

$$\mathbf{a}t^\Lambda = \text{diag}(a_1 t^{\Lambda_1}, a_2 t^{\Lambda_2}, \dots, a_N t^{\Lambda_N}). \tag{4.21}$$

Recall that the powers of t comes from the necessity to preserve the boundary profile (4.7). Here we need to make the action of the outer automorphism \mathcal{P}_σ compatible with it. Namely, we pick an inner action B_σ such that

$$B_\sigma \mathcal{P}_\sigma(\Lambda(\tau_i)) B_\sigma^{-1} = \Lambda(\tau_i) , \tag{4.22}$$

where τ_i are the standard generators of $\text{SU}(2)$. For τ_3 , this can be achieved by making $\Lambda_i = -\Lambda_{N+1-i}$ (for $i = 1, \dots, N$) by reordering. For $\tau_\pm = \tau_1 \pm i\tau_2$, the equation above can be solved by a diagonal B_σ with $\pm 1, \pm i$ as entries.

Therefore, the rule to obtain the argument of the orthogonal polynomials becomes the following:

1. Restrict the flavor fugacities so that we have

$$\begin{aligned}
&\text{for } N = 2l, \quad a_i = a_{N+1-i}^{-1} , \quad \forall i = 1, \dots, l ; \\
&\text{for } N = 2l + 1, \quad a_i = a_{N+1-i}^{-1} , \quad a_{l+1} = 1 \quad \forall i = 1, \dots, l .
\end{aligned} \tag{4.23}$$

2. Multiplies with factors of $\pm 1, \pm i$ coming from B_σ .
3. Select half of the elements from the tuple, one from each pair exchanged by \mathbb{Z}_2 action. As a convention we choose those whose powers in t are non-negative.

For example,

- For $\Lambda_{\text{full}} = [1^N]$, i.e. the full puncture, the profile $\Lambda(\tau_i) = 0$ automatically satisfies (4.22). So we have

$$\mathbf{a}t^\Lambda = (a_1, a_2, \dots, a_l) , \quad l = \lfloor N/2 \rfloor . \tag{4.24}$$

- For $\rho = \Lambda = [N]$, i.e. the null puncture, the profile

$$\Lambda(\tau_+) = \begin{cases} c_0 e_{-1,1} + \sum_{s=1}^{l-1} c_s (e_{-s-1,-s} + e_{s,s+1}), & N = 2l \\ \sum_{s=0}^{l-1} c_s (e_{-s-1,-s} + e_{s,s+1}), & N = 2l + 1 \end{cases} \tag{4.25}$$

is invariant under the action \mathcal{P}_σ given in (2.15), (2.17). For example,

$$\Lambda(\tau_+) = \begin{pmatrix} 0 & \bullet & 0 & 0 \\ 0 & 0 & \bullet & 0 \\ 0 & 0 & 0 & \bullet \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (4.26)$$

for $N = 4$; see the right diagram in Fig. 1. We have

$$\mathbf{a}t^\Lambda = t^\rho = \begin{cases} (t^{l-1/2}, t^{l-3/2}, \dots, t^{1/2}) , & N = 2l \\ (t^l, t^{l-1}, \dots, t) , & N = 2l + 1 . \end{cases} \quad (4.27)$$

– For $\Lambda_{\text{simple}} = [N - 1, 1]$, i.e. the simple puncture, the profile

$$\Lambda(\tau_+) = \begin{cases} c'_0(e_{-2,1} + e_{-1,2}) + \sum_{s=1}^{l-1} c'_s(e_{-s-1,-s} + e_{s,s+1}), & N = 2l \\ c'_0 e_{-1,1} + \sum_{s=1}^{l-1} c'_s(e_{-s-1,-s} + e_{s,s+1}), & N = 2l + 1 \end{cases} \quad (4.28)$$

is *not* invariant under the action \mathcal{P}_σ given in (2.17), (2.15). For example,

$$\Lambda(\tau_+) = \begin{pmatrix} 0 & \bullet & \bullet & 0 \\ 0 & 0 & 0 & \bullet \\ 0 & 0 & 0 & \bullet \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (4.29)$$

for $N = 4$, and compare it with the right diagram of Fig. 1. This non-invariance can be canceled by

$$B_\sigma = \begin{cases} \text{diag}(\underbrace{i, \dots, i}_l, \underbrace{-i, \dots, -i}_l), & N = 2l \\ \text{diag}(\underbrace{i, \dots, i}_l, 1, \underbrace{-i, \dots, -i}_l), & N = 2l + 1. \end{cases} \quad (4.30)$$

Then we have

$$t^{\Lambda_{\text{simple}}} = \begin{cases} (it^{l-1}, it^{l-2}, \dots, i) , & N = 2l \\ (it^{l-1/2}, it^{l-1/2}, \dots, it^{1/2}) , & N = 2l + 1 . \end{cases} \quad (4.31)$$

More examples are discussed in the following sections.

- In the untwisted case, the prefactor $K_\Lambda(\mathbf{a})$ is given by the plethystic exponential (4.11), i.e. by products of infinite products of the form

$$K_\Lambda(\mathbf{a}) = \prod_s \prod_{n=0} \frac{1}{1 - z_{\Lambda,s}(\mathbf{a}, q, t) q^n} , \quad (4.32)$$

where $z_s(q, t, a)$ are *monomials* of q, t and the fugacities a . Each choice of the index s corresponds to a basis of the Slodowy slice, and generates the Fock space of a Kaluza-Klein tower with the fugacity z_s and the mode number n . The \mathbb{Z}_2 twist assigns a \pm sign to each of the factors, resulting in

$$\tilde{K}_{\Lambda}(\mathbf{a}) = \prod_s \prod_{n=0} \frac{1}{1 \pm z_{\Lambda,s}(\tilde{\mathbf{a}}, q, t) q^n} , \quad (4.33)$$

where ± 1 is determined by the action of $B_{\sigma} \mathcal{P}_{\sigma}$ on the Slodowy slice. For example,

- when $\Lambda = [1^N]$, the untwisted prefactor $K_{\Lambda}(\mathbf{a})$ was the conversion factor between the vector multiplet measure and the Macdonald measure, as discussed in (4.12), (4.13). The twisted prefactor is also the conversion factor between them:

$$\tilde{K}_{\text{full}}(\mathbf{z}) = \begin{cases} \frac{(-t; q)}{(t; q)^l (-t; q)^l} \text{PE} \left[\frac{t}{1-q} E_1(\mathbf{z}) + \frac{t^2}{1-q^2} E_2(\mathbf{z}) \right] , & N = 2l \\ \frac{(-t, q)}{(-t; q)^{l+1} (t; q)^l} \text{PE} \left[\frac{ut}{1-q} E_1(\mathbf{z}) + \frac{t}{1-q} E_2(\mathbf{z}) + \frac{t^2}{1-q^2} E_1(\mathbf{z}) \right]_{u=-1} , & N = 2l + 1 . \end{cases} \quad (4.34)$$

- when $\Lambda = [N]$, we have

$$\tilde{K}_{[N]} = \tilde{K}_{\rho} = \prod_{i=2}^N \frac{1}{((-t)^i; q)} . \quad (4.35)$$

The general rules for the twisted prefactors are given in Appendix C. More examples can be found in the following sections.

Let us now move on to the examples where we can test the formula explained above.

4.2.1 Free bifundamental hypermultiplets

The two-parameter index of a hypermultiplet in the representation R without outer-automorphism twist is given by the formula

$$\text{PE} \left[\frac{t^{1/2}}{1-q} \chi_{R \oplus \bar{R}}(\mathbf{x}) \right] . \quad (4.36)$$

In particular, for a bifundamental hypermultiplet of $\text{SU}(N) \times \text{SU}(N)$, we have

$$\mathcal{I}_N^{\text{bifund}}(q, t, \mathbf{x}, \mathbf{y}) = \text{PE} \left[\frac{t^{1/2}}{1-q} \left\{ \chi_{\text{fund}}^{\text{SU}(N)}(\mathbf{x}) \chi_{\text{fund}}^{\text{SU}(N)}(\mathbf{y}^{-1}) + \chi_{\text{fund}}^{\text{SU}(N)}(\mathbf{x}^{-1}) \chi_{\text{fund}}^{\text{SU}(N)}(\mathbf{y}) \right\} \right] . \quad (4.37)$$

As was the case for vector multiplet contributions studied in Sec. 3.2, this infinite product comes from the trace over the Fock space generated by the raising operators

$$v_{i,j;n}, \quad w_{i,j;n} \quad (4.38)$$

which have the charge $x_i/y_j t^{1/2} q^n$ and $y_j/x_i t^{1/2} q^n$, respectively. From the charge assignments, it is clear that the \mathbb{Z}_2 operation just exchanges the modes $v_{i,j;n}$ and $w_{i,j;n}$. The superpotential term contains a term of the form $\sum_i e_{i,i} v_{i,j;n} w_{i,j;n}$ where $e_{i,i}$ denotes a mode in the vector multiplet with the specified gauge indices. This should be invariant under the \mathbb{Z}_2 action, which acts on $e_{i,i}$ via $e_{i,i} \rightarrow -e_{-i,-i}$. Therefore we find that the \mathbb{Z}_2 action on $v_{i,j;n}$ and $w_{i,j;n}$ needs to have the form

$$v_{i,j;n} \rightarrow w_{i,j;n}, \quad w_{i,j;n} \rightarrow -v_{i,j;n}. \quad (4.39)$$

To perform the \mathbb{Z}_2 operation inside the trace, one now needs to do the following replacements:

$$\begin{aligned} \text{For } N = 2l, \quad x_i &= x_{N+1-i}^{-1}, \quad y_i = y_{N+1-i}^{-1}, \quad \forall i = 1, \dots, l, \\ \text{For } N = 2l + 1, \quad x_i &= x_{N+1-i}^{-1}, \quad y_i = y_{N+1-i}^{-1}, \quad x_{l+1} = y_{l+1} = 1 \quad \forall i = 1, \dots, l. \end{aligned} \quad (4.40)$$

We can therefore obtain the twisted character of the bifundamentals in two steps: (1) performing the replacements (4.40), and (2) combining of two modes to get a \mathbb{Z}_2 invariant combination (“doubling”). The process can be explicitly written down as follows:

$$\begin{aligned} & \chi_{\text{fund}}^{\text{SU}(N)}(\mathbf{x}) \chi_{\text{fund}}^{\text{SU}(N)}(\mathbf{y}^{-1}) + \chi_{\text{fund}}^{\text{SU}(N)}(\mathbf{x}^{-1}) \chi_{\text{fund}}^{\text{SU}(N)}(\mathbf{y}) \\ & \xrightarrow{\text{Step (1)}} \begin{cases} 2E_1(\mathbf{x}^{1/2})E_1(\mathbf{y}^{1/2}), & N = 2l \\ 2(E_1(\mathbf{x}^{1/2}) + 1)(E_1(\mathbf{y}^{1/2}) + 1), & N = 2l + 1 \end{cases} \\ & \xrightarrow{\text{Step (2)}} \begin{cases} E_1(\mathbf{x})E_1(\mathbf{y}), & N = 2l \\ (E_1(\mathbf{x}) + 1)(E_1(\mathbf{y}) + 1), & N = 2l + 1. \end{cases} \end{aligned} \quad (4.41)$$

Thus, the superconformal index is given by

$$\tilde{\mathcal{I}}_N^{\text{bifund}} = \begin{cases} \text{PE} \left[\frac{ut}{1-q} E_1(\mathbf{x}) E_1(\mathbf{y}) \right]_{u=-1}, & N = 2l \\ \text{PE} \left[\frac{ut}{1-q} (E_1(\mathbf{x}) + 1)(E_1(\mathbf{y}) + 1) \right]_{u=-1}, & N = 2l + 1. \end{cases} \quad (4.42)$$

Note that we used fermionic plethystic exponent since we get the minus sign in the index from (4.39) for a paired up mode $v_{i,j;n} w_{i,j;n}$.

Let us next consider the three-punctured spheres realizing the free bifundamental hypermultiplet, which is a sphere with two full punctures and one simple puncture. The arguments of Macdonald polynomials are already discussed, see (4.27) and (4.31). The prefactor $\tilde{K}_{\text{simple}}$ is determined to be

$$\tilde{K}_{[2l-1,1]} = \frac{1}{(t^{2l}; q^2)} \prod_{i=1}^{2l-1} \frac{1}{((-t)^i; q)}, \quad \tilde{K}_{[2l,1]} = \frac{1}{(-t^{2l+1}; q^2)} \prod_{i=1}^{2l} \frac{1}{((-t)^i; q)}. \quad (4.43)$$

The general formula then gives the twisted index

$$\tilde{\mathcal{I}}^{\text{bifund}} = \frac{\tilde{K}_{\text{full}}(\mathbf{x}) \tilde{K}_{\text{full}}(\mathbf{y}) \tilde{K}_{\text{simple}}}{\tilde{K}_{\rho}} \sum_{\lambda} \frac{\tilde{P}_{\lambda}(\mathbf{x}) \tilde{P}_{\lambda}(\mathbf{y}) \tilde{P}_{\lambda}(it^{\Lambda_{\text{simple}}})}{\tilde{P}_{\lambda}(t^{\rho})}. \quad (4.44)$$

where \tilde{K}_{full} , \tilde{K}_{ρ} are given by (4.34), (4.35), respectively. We checked the agreement of it with (4.42) for the cases of $N = 2, 3, 4, 5, 6, 7$ by expanding both sides in series in q, t by Mathematica. The twisted superconformal index of other three-punctured spheres which give free hypermultiplets are discussed in Appendix D.

4.2.2 The Argyres-Seiberg duality

In [22], the superconformal index of the T_3 theory has been computed using the inverse formula of certain elliptic beta integral [23]. From the Argyres-Seiberg duality [24], $\text{SU}(3)$ theory with $N_f = 6$ is dual to E_6 SCFT with $\text{SU}(2)$ flavor subgroup gauged and coupled to a fundamental hypermultiplet. Schematically, the superconformal index of the theory can be written as

$$\begin{aligned}\mathcal{I}(a, \mathbf{x}, b, \mathbf{y}) &= \int [d\mathbf{z}]_{\text{SU}(3)} \mathcal{I}_{\text{SU}(3)}^{\text{hyp}}(a, \mathbf{x}; \mathbf{z}) \mathcal{I}_{\text{SU}(3)}^V(\mathbf{z}) \mathcal{I}_{\text{SU}(3)}^{\text{hyp}}(b, \mathbf{y}; \mathbf{z}) \\ &= \int [ds]_{\text{SU}(2)} \mathcal{I}_{E_6}(\mathbf{x}; \mathbf{y}; a, s) \mathcal{I}_{\text{SU}(2)}^V(s) \mathcal{I}_{\text{SU}(2)}^{\text{hyp}}(s; b),\end{aligned}\quad (4.45)$$

where a, b are $\text{U}(1)$ flavor fugacities and $\mathbf{x}, \mathbf{y}, \mathbf{z}$ are the $\text{SU}(3)$ flavor fugacities. Let's explain the physical meaning of each line briefly.

- In the first line, we start from two identical theories, each containing the $\text{SU}(3) \times \text{SU}(3)$ free bifundamentals. One of the $\text{SU}(3)$ global symmetries from each theory is then gauged and coupled together. The fugacities \mathbf{z} correspond to the gauged $\text{SU}(3)$ group and $\mathcal{I}_{\text{SU}(3)}^V(\mathbf{z})$ is the corresponding vector multiplet index.
- In the second line, we decompose the E_6 global symmetry into its maximal subgroup $\text{SU}(3)^3$. We denote by \mathbf{x} and \mathbf{y} the fugacities of two of these $\text{SU}(3)$. The remaining $\text{SU}(3)$ global symmetry is decomposed further into its subgroup $\text{SU}(2) \times \text{U}(1)$, whose fugacities are denoted by s and a respectively. This $\text{SU}(2)$ is then gauged and coupled to the theory with a free bifundamental in $\text{SU}(2) \times \text{U}(1)$, whose fugacities are denoted by s and b respectively. Here $\mathcal{I}_{\text{SU}(2)}^V(s)$ is the vector multiplet index corresponding to the gauged $\text{SU}(2)$ group.

The same kind of relation should hold for the twisted index due to the duality. Even though there is no analogous inversion formula for the twisted index, we can use the TQFT structure of the index to write down a natural conjecture for the index [2, 3]. In this section, we will conjecture such a twisted index for the E_6 SCFT and then verify it against the Argyres-Seiberg duality.

Let's first start with the case with the weakly coupled frame. The vector multiplet twisted index for the $\text{SU}(3)$ theory is given by the integrand of (3.17) with $(r' = 0, r'' = 1)$:

$$\tilde{\mathcal{I}}^V(x) = \text{PE} \left[\frac{-q^2 - t^2}{1 - q^2} \left(x^2 + \frac{1}{x^2} + 1 \right) + \frac{-uq - ut}{1 - q} \left(x^2 + \frac{1}{x^2} \right) \right]_{u=-1}, \quad (4.46)$$

and the hypermultiplet contribution is given by (4.42):

$$\tilde{\mathcal{I}}^{\text{hyp}}(z; x) = \text{PE} \left[\frac{ut}{1-q^2} \left(x^2 + \frac{1}{x^2} + 1 \right) \left(z^2 + \frac{1}{z^2} + 1 \right) \right]_{u=-1}. \quad (4.47)$$

The twisted index for the SU(3) theory with $N_f = 6$ is now given by

$$\tilde{\mathcal{I}}_{\text{weak}}(z, y) = \oint_{|x|=1} \frac{dx}{2\pi i x} \tilde{\Delta}_{A_2}(x) \tilde{\mathcal{I}}^V(x) \tilde{\mathcal{I}}^{\text{hyp}}(z; x^{-1}) \tilde{\mathcal{I}}^{\text{hyp}}(y; x). \quad (4.48)$$

where

$$\tilde{\Delta}_{A_2}(x) = \Delta_{C_1}(x) = \frac{1}{2}(1-x^4)(1-x^{-4}) \quad (4.49)$$

is the Haar measure of C_1 .

Now, in the dual strongly coupled frame, the twisted index should be written as

$$\tilde{\mathcal{I}}_{\text{strong}}(z; y) = \oint_{|x|=1} \frac{dx}{2\pi i x} \tilde{\Delta}(x) \tilde{\mathcal{I}}_{\text{SU}(2)}^V(x) \tilde{\mathcal{I}}_{\text{SU}(2)}^{\text{hyp}}(x) \tilde{\mathcal{I}}_{E_6}(x, y, z). \quad (4.50)$$

Here the important thing is that the twisted Haar measure $\tilde{\Delta}(x)$ is not that of C_1 as one would naively expect. Note that we need to decompose 3 of SU(3) into $2 \oplus 1$ of SU(2) which should be compatible with the σ action. Since σ acts on SU(3) by sending three indices 1, 0, -1 to -1, 0, 1 (see (4.23)), the SU(2) subgroup compatible with this action is the subgroup of matrices of the form:

$$\begin{pmatrix} a & 0 & e_{1,-1} \\ 0 & 0 & 0 \\ e_{-1,1} & 0 & -a \end{pmatrix}. \quad (4.51)$$

Since σ maps $e_{i,j}$ to $-e_{-j,-i}$, the SU(2) subgroup given above keeps only the modes $e_{1,-1}$ and $e_{-1,1}$ out of the six roots of SU(3). The twisted Haar measure is thus given by

$$\tilde{\Delta}(x) = \frac{1}{2}(1+x^2) \left(1 + \frac{1}{x^2} \right). \quad (4.52)$$

Note the sign difference from the Haar measure (4.49) of C_1 ; the plus signs in (4.52) are due to the minus sign in the action of \mathcal{P}_σ in (2.21).

The vector multiplet contribution is not also the same as the twisted SU(2) or SU(3). Since the gauge group is coming from gauging the $\text{SU}(2) \subset \text{SU}(3)$, we find

$$\tilde{\mathcal{I}}_{\text{SU}(2)}^V(x) = \text{PE} \left[\frac{-q-t}{1-q} \right] \text{PE} \left[\frac{-uq-ut}{1-q} \left(x^2 + \frac{1}{x^2} \right) \right]_{u=-1}. \quad (4.53)$$

The first term comes from the Cartan of SU(2) and the second term comes from the states built out of roots $e_{\pm 1, \mp 1}$. The hypermultiplet contribution is simply given by (4.42):

$$\tilde{\mathcal{I}}_{\text{SU}(2)}^{\text{hyp}}(x) = \text{PE} \left[\frac{ut}{1-q^2} \left(x^2 + \frac{1}{x^2} \right) \right]_{u=-1}. \quad (4.54)$$

From the TQFT structure, suppose the twisted index of Minahan-Nemeschansky E_6 theory is given by a 3-point function for the 3 full-punctures of the form

$$\tilde{\mathcal{I}}_{E_6}(x, y, z) = \tilde{K}_{\text{full}}(x)\tilde{K}_{\text{full}}(y)\tilde{K}_{\text{full}}(z) \sum_{\lambda} \frac{\tilde{P}_{\lambda}(x)\tilde{P}_{\lambda}(y)\tilde{P}_{\lambda}(z)}{\tilde{K}_{\rho}\tilde{P}_{\lambda}(t^{\rho})}. \quad (4.55)$$

The prefactors $\tilde{K}_{\text{full}} = \tilde{K}_{[1,1,1]}$ and \tilde{K}_{ρ} are given by (4.34) and (4.35) respectively, and $\tilde{P}_{\lambda}(x)$ is the Askey-Wilson polynomial or the Macdonald polynomial of type (C_1^{\vee}, C_1) .

The strongly coupled frame answer is given by

$$\tilde{\mathcal{I}}_{\text{strong}} = \int \frac{dx}{2\pi i x} \tilde{\Delta}(\mathbf{x}) \tilde{\mathcal{I}}_{E_6}(x, y, z) \tilde{\mathcal{I}}_{\text{SU}(2)}^V(x) \tilde{\mathcal{I}}_{\text{SU}(2)}^{\text{hyp}}(x). \quad (4.56)$$

We find

$$\tilde{\mathcal{I}}_{\text{strong}} = \tilde{\mathcal{I}}_{\text{weak}} \quad (4.57)$$

as expected from the Argyres-Seiberg duality. Of course it can also be expressed by the general formula (4.17) applied to the four-punctured sphere with two full punctures and two simple punctures.

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A Study of the 2d propagator

Let us consider the vector multiplet measure from the point of view in [25]. The 4d theory obtained by compactifying the 6d theory of type Γ on a cylinder with finite area is a $\mathcal{N} = 2$ non-linear sigma model whose target space is $T^*\Gamma_{\mathbb{C}}$. Here and in the following we use Γ to denote the group. Its partition function on $S^3 \times S^1$ boils down essentially to a partition function of a quantum mechanics on the group manifold Γ . By the theorem of Peter-Weyl, the wave functions on the group manifold Γ has the irreducible decomposition under the $\Gamma \times \Gamma$ action as

$$\text{functions on } \Gamma \simeq \bigoplus_{\lambda} R_{\lambda} \otimes \bar{R}_{\lambda}, \quad (\text{A.1})$$

where R_λ is the irrep with the highest weight λ . Then the trace of the chemical potentials (\mathbf{x}, \mathbf{y}) for $\Gamma \times \Gamma$ is

$$\sum_{\lambda} \chi_{\lambda}(\mathbf{x}) \overline{\chi_{\lambda}(\mathbf{y})} = \sum_{\lambda} \chi_{\lambda}(\mathbf{x}) \chi_{\lambda}(\mathbf{y}^{-1}). \quad (\text{A.2})$$

This is the properly-normalized delta-function for the Haar measure (2.22) of Γ , and is the 2d Yang-Mills propagator

Now let us say Γ is simply-laced, and include the action of an outer-automorphism σ on Γ . The outer-automorphism σ can send R_λ to a different representation $R_{\lambda'}$, in which case R_λ doesn't contribute to the trace anymore, or R_λ is sent to itself R_λ , by an action \mathcal{P}_σ on R_λ , in which case R_λ contributes to the trace by $\text{tr}_{R_\sigma} \mathcal{P}_\sigma x$. Then the trace is

$$\sum_{\lambda=\sigma(\lambda)} (\text{tr}_{R_\lambda} \mathcal{P}_\sigma \mathbf{x}) (\text{tr}_{R_\lambda} \mathcal{P}_\sigma \mathbf{y}^{-1}). \quad (\text{A.3})$$

Now, in [26] it was proved that

- the set $\lambda = \sigma(\lambda)$ of the irreps of Γ is naturally identified with the set $\underline{\lambda}$ of the irreps of G , and
- $\text{tr}_{R_\lambda} \mathcal{P}_\sigma \mathbf{x} = \underline{\chi}_{\underline{\lambda}}(\mathbf{x})$ under this correspondence, where $\underline{\chi}_{\underline{\lambda}}$ is the character of G of the irrep $\underline{R}_{\underline{\lambda}}$.

Then we conclude that (A.3) is equal to

$$\sum_{\underline{\lambda}} \underline{\chi}_{\underline{\lambda}}(\mathbf{x}) \underline{\chi}_{\underline{\lambda}}(\mathbf{y}^{-1}), \quad (\text{A.4})$$

which is the propagator of the 2d Yang-Mills of non-simply-laced G .

B Computations of Macdonald polynomials

We want to explicitly find the Macdonald polynomials P_λ that are orthogonal with respect to the Macdonald measure. The Macdonald polynomial has a feature that

$$P_\lambda(\mathbf{a}; q, t) = \chi_\lambda(\mathbf{a}) + \sum_{\mu < \lambda} f_\mu(q, t) \chi_\mu(\mathbf{a}), \quad (\text{B.1})$$

where the sum over μ is only for $\mu < \lambda$, in the sense that

$$\lambda - \mu = \text{a sum of positive number of simple roots.} \quad (\text{B.2})$$

To achieve this, we just have to do the Schmidt orthogonalization with respect to the Macdonald measure, starting from

$$\chi_{\lambda_1}, \chi_{\lambda_2}, \dots \quad (\text{B.3})$$

such that $\lambda_i < \lambda_j$ whenever $i < j$.

In practice, to obtain such an ordered set, one can compute the inner products $w_i = \lambda_i \cdot \rho$, where ρ is the Weyl vector of C_ℓ , and perform the sorting

$$\chi_{w_1}, \chi_{w_2}, \dots \quad (\text{B.4})$$

such that $w_i < w_j$ whenever $i < j$.

Also, for the Macdonald polynomials of type (C_1^\vee, C_1) (which are also called the 1-variable Koornwinder polynomials, or the Askey-Wilson polynomials) we can use the explicit formula in terms of q -hypergeometric functions [27].

C The rule for the prefactor $\tilde{K}_\Lambda(a)$

Consider a puncture of type $[l_1, l_2, \dots, l_k]$ such that $\sum l_i = N$. Denote the original mass parameters by a_1, a_2, \dots such that $a_1^{l_1} a_2^{l_2} \dots a_k^{l_k} = 1$. To twist by \mathbb{Z}_2 , we need to further impose the constraint that

$$(a_1 t^{\frac{l_1-1}{2}}, a_1 t^{\frac{l_1-3}{2}}, \dots, a_1 t^{\frac{1-l_1}{2}}, a_2 t^{\frac{l_2-1}{2}}, \dots, a_2 t^{\frac{1-l_2}{2}}, \dots, a_k t^{\frac{l_k-1}{2}}, \dots, a_k t^{\frac{1-l_k}{2}}). \quad (\text{C.1})$$

is invariant under the \mathbb{Z}_2 operation. We also need to include the factors of ± 1 and $\pm i$ coming from the matrix B_σ in (4.22). Then, the prefactor is given by multiplying the following factors:

- An overall factor of $(-q; q)$.
- For each $i < j$ such that $m = l_i \neq n = l_j$,

$$\text{PE} \left[u^{0,1} \frac{t^{m-n+2} + t^{m-n+4} + \dots + t^{m+n}}{1 - q^2} \left(\frac{a_m^2}{a_n^2} + \frac{a_n^2}{a_m^2} \right) \right]_{u=-1} \quad (\text{C.2})$$

Here $u^{0,1}$ stands for either u or 1 .

- Say the columns from i to j all have height m .
 - When the number of columns is even, $i - j + 1 = 2\ell$. Call the mass parameters $z_1^{\pm 1}, \dots, z_\ell^{\pm 1}$.

$$\frac{1}{\prod_{n=1}^m ((-t)^n; q)^\ell (-(-t)^n; q)^\ell} \text{PE} \left[\frac{t^2 + t^4 + \dots + t^{2m}}{1 - q^2} E_2(\mathbf{z}) + u^{0,1} \frac{ut + (ut)^2 + \dots + (ut)^m}{1 - q} E_1(\mathbf{z}) \right] \quad (\text{C.3})$$

- When the number of columns is odd, $i - j + 1 = 2\ell + 1$. Do include the case $\ell = 0$. Call the mass parameters $z_1^{\pm 1}, \dots, z_\ell^{\pm 1}, 1$.

$$\frac{1}{\prod_{n=1}^m ((-t)^n; q)^{\ell+1} (-(-t)^n; q)^\ell} \text{PE} \left[\begin{aligned} & + \frac{t^2 + t^4 + \dots + t^{2m}}{1 - q^2} E_2(\mathbf{z}) + \frac{t^2 + t^4 + \dots + t^{2m}}{1 - q^2} E_1(\mathbf{z}) \\ & + u^{0,1} \frac{ut + (ut)^2 + \dots + (ut)^m}{1 - q} E_1(\mathbf{z}) \end{aligned} \right] . \quad (\text{C.4})$$

It should be possible to determine the signs by carefully studying the action of \mathcal{P}_σ and B_σ on the Slodowy slices.

D Other free hypermultiplets

D.1 A sphere with punctures $[2, 2]$, $[2, 1, 1]$ and $[1, 1, 1, 1]$

Without the twisting, in terms of chiral superfields, we have matter in the following representations of $\text{SU}(2) \times \text{SU}(2) \times \text{SU}(4)$: $[1; 0; 1, 0, 0]$, $[1; 0; 0, 0, 1]$ and $[0; 1; 0, 1, 0]$, see [28, 29].

The only possible \mathbb{Z}_2 action exchanges $\text{SU}(4)$ representation $[1, 0, 0]$ with $[0, 0, 1]$. Up to this point, we are still keeping all the fugacities of $\text{SU}(4)$, namely z_1, z_2, z_3 . Note the characters of the following representations

$$\chi_{[1,0,0]}^{A_3}(\mathbf{z}) = \sum_{i=1}^3 z_i, \quad \chi_{[0,0,1]}^{A_3}(\mathbf{z}) = \sum_{i=1}^3 z_i^{-1}, \quad \chi_{[0,1,0]}^{A_3}(\mathbf{z}) = \sum_{1 \leq i < j \leq 3} z_i z_j. \quad (\text{D.1})$$

The \mathbb{Z}_2 operation thus sends $z_i \rightarrow 1/z_i$. Next, we set

$$z_4 = z_1^{-1}, \quad z_3 = z_2^{-1}. \quad (\text{D.2})$$

The twisted character for the chiral fields in $[1; 0; 1, 0, 0] + [1; 0; 0, 0, 1]$ of $\text{SU}(2) \times \text{SU}(2) \times \text{SU}(4)$ can be computed using the replacement (D.2) and taking into account of the doubling; we have

$$\begin{aligned} \chi_{[1]}^{A_1}(\mathbf{x}) \chi_{[1,0,0]}^{A_3}(\mathbf{z}) + \chi_{[1]}^{A_1}(\mathbf{x}) \chi_{[0,0,1]}^{A_3}(\mathbf{z}) &= (x + x^{-1}) \sum_{i=1}^3 (z_i + z_i^{-1}) \\ \rightarrow (x^2 + x^{-2}) \sum_{i=1}^3 (z_i^2 + z_i^{-2}) &= (x^2 + x^{-2}) E_1(\mathbf{z}). \end{aligned} \quad (\text{D.3})$$

Let us now consider the chiral fields in $[0; 1; 0, 1, 0]$ of $\text{SU}(2) \times \text{SU}(2) \times \text{SU}(4)$. Under the replacement (D.2) (without any doubling at this stage), we have

$$\begin{aligned} \chi_{[1]}^{A_1}(\mathbf{y}) \chi_{[0,1,0]}^{A_3}(\mathbf{z}) &\rightarrow 2(y + y^{-1}) + (y + y^{-1}) \sum_{\epsilon_i = \pm 1} z_1^{\epsilon_1} z_2^{\epsilon_2} \\ &= 2(y + y^{-1}) + (y + y^{-1}) \widehat{E}_2(\mathbf{z}). \end{aligned} \quad (\text{D.4})$$

Since the representation $[0, 1, 0]$ of $SU(4)$ is strictly real, there is no doubling for the second term $(y + y^{-1})\widehat{E}_2(\mathbf{z})$ of (D.4). However, there is still a doubling for the first term $2(y + y^{-1})$ of (D.4); this leads to $(y^2 + y^{-2})$.

Thus, we have the twisted index

$$\begin{aligned} \widetilde{\mathcal{I}}_{[2,2],[2,1,1],[1^4]} = \text{PE} \left[\frac{ut}{1-q^2}(x^2 + x^{-2})E_1(\mathbf{z}) + \frac{t}{1-q^2}(y^2 + y^{-2}) \right. \\ \left. + \frac{ut^{1/2}}{1-q}(y + y^{-1})\widehat{E}_2(\mathbf{z}) \right]_{u=-1}. \end{aligned} \quad (\text{D.5})$$

This can be written in the following form

$$\widetilde{\mathcal{I}}_{[2,2],[2,1,1],[1^4]} = \frac{\widetilde{K}_{[2,2]}\widetilde{K}_{[2,1,1]}\widetilde{K}_{\text{full}}}{\widetilde{K}_{\rho}} \sum_{\lambda} \frac{\widetilde{P}_{\lambda}(\mathbf{x})\widetilde{P}_{\lambda}(\mathbf{y})\widetilde{P}_{\lambda}(\mathbf{z})}{\widetilde{P}_{\lambda}(t^{\rho})}, \quad (\text{D.6})$$

where \widetilde{K}_{Λ} , \mathbf{x} and \mathbf{y} are given below.

D.1.1 Puncture $[2, 2]$

For the puncture $[2, 2]$, the untwisted fugacity assignment is $(t^{1/2}x, t^{1/2}/x, t^{-1/2}x, t^{-1/2}/x)$. Therefore, the profile $\Lambda(\tau_+)$ has the form

$$\Lambda(\tau_+) = \begin{pmatrix} 0 & 0 & \bullet & 0 \\ 0 & 0 & 0 & \bullet \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0. \end{pmatrix} \quad (\text{D.7})$$

Comparing with (2.15) and Fig. 1, we find

$$B_{\sigma} = \text{diag}(i, i, -i, -i). \quad (\text{D.8})$$

Then the argument of \widetilde{P} after the twist is

$$\mathbf{x} = (ixt^{1/2}, it^{1/2}/x) \quad (\text{D.9})$$

while the prefactor is

$$\widetilde{K}_{[2,2]}(\mathbf{x}) = \frac{(-t; q)}{\{(-t; q)(t; q)\}\{(t^2; q)(-t^2; q)\}} \text{PE} \left[\left(\frac{t}{1-q} + \frac{ut^2}{1-q^2} \right) (x^2 + x^{-2}) \right]_{u=-1}. \quad (\text{D.10})$$

D.1.2 Puncture $[2, 1, 1]$

For the puncture $[2, 1, 1]$, the untwisted fugacity assignment is $(t^{1/2}, y, 1/y, t^{-1/2})$. Therefore, the profile $\Lambda(\tau_+)$ has the form

$$\Lambda(\tau_+) = \begin{pmatrix} 0 & 0 & 0 & \bullet \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (\text{D.11})$$

Comparing with (2.15) and Fig. 1, we find that B_σ is trivial. Then the argument of \tilde{P} is

$$\mathbf{y} = (y, t^{1/2}) \quad (\text{D.12})$$

while the prefactor is

$$\tilde{K}_{[2,1,1]}(\mathbf{y}) = \frac{(-t; q)}{\{(-t; q)(t; q)\}\{(-t; q)(t^2; q)\}} \text{PE} \left[\left(\frac{t}{1-q} + \frac{t^3}{1-q^2} \right) (y^2 + y^{-2}) \right]. \quad (\text{D.13})$$

D.2 A sphere with punctures $[3, 2]$, $[2, 2, 1]$ and $[1, 1, 1, 1, 1]$

Without the twisting, in terms of chiral superfields, we have matter in the following representations of $\text{SU}(2) \times \text{SU}(5)$: $[1; 1, 0, 0, 0]$, $[1; 0, 0, 0, 1]$, $[0; 0, 1, 0, 0]$ and $[0; 0, 0, 1, 0]$.

The only possible \mathbb{Z}_2 action exchanges $[0, 1, 0, 0]$ with $[0, 0, 1, 0]$, and $[1, 0, 0, 0]$ with $[0, 0, 0, 1]$. Up to this point, we are still keeping all the fugacities of $\text{SU}(5)$, namely z_1, \dots, z_5 . Note the characters of the following representations

$$\begin{aligned} \chi_{[1,0,0,0]}^{A_4}(\mathbf{z}) &= \sum_{i=1}^5 z_i^1, & \chi_{[0,0,0,1]}^{A_4}(\mathbf{z}) &= \sum_{i=1}^5 z_i^{-1}, \\ \chi_{[0,1,0,0]}^{A_4}(\mathbf{z}) &= \sum_{1 \leq i < j \leq 5} z_i z_j, & \chi_{[0,0,1,0]}^{A_4}(\mathbf{z}) &= \sum_{1 \leq i < j \leq 5} z_i^{-1} z_j^{-1}. \end{aligned} \quad (\text{D.14})$$

The \mathbb{Z}_2 operation thus sends $z_i \rightarrow 1/z_i$. Next, we set

$$z_5 = z_1^{-1}, \quad z_4 = z_2^{-1}, \quad z_3 = 1. \quad (\text{D.15})$$

Hence, under the replacement (D.15) and taking into account of the doubling, we have

$$\begin{aligned} \chi_{[1]}^{A_1}(\mathbf{y}) \chi_{[1,0,0,0]}^{A_5}(\mathbf{z}) + \chi_{[1]}^{A_1}(\mathbf{y}) \chi_{[0,0,0,1]}^{A_5}(\mathbf{z}) &\rightarrow (y^2 + y^{-2})(E_1(\mathbf{z}) + 1), \\ \chi_{[0,1,0,0]}^{A_5}(\mathbf{z}) + \chi_{[0,0,1,0]}^{A_5}(\mathbf{z}) &\rightarrow E_1(\mathbf{z}) + E_2(\mathbf{z}) + 1. \end{aligned} \quad (\text{D.16})$$

The twisted index is given by

$$\tilde{\mathcal{I}}_{[3,2],[2,2,1],[1^5]} = \text{PE} \left[\frac{ut}{1-q} \{ (y^2 + y^{-2})(E_1(\mathbf{z}) + 1) + E_1(\mathbf{z}) + E_2(\mathbf{z}) + 1 \} \right]_{u=-1}. \quad (\text{D.17})$$

This can be written in the following form

$$\tilde{\mathcal{I}}_{[3,2],[2,2,1],[1^5]} = \frac{\tilde{K}_{[3,2]} \tilde{K}_{[2,2,1]} \tilde{K}_{\text{full}}}{\tilde{K}_\rho} \sum_{\lambda} \frac{\tilde{P}_\lambda(\mathbf{x}) \tilde{P}_\lambda(\mathbf{y}) \tilde{P}_\lambda(\mathbf{z})}{\tilde{P}_\lambda(t^\rho)}, \quad (\text{D.18})$$

where \tilde{K}_Λ , \mathbf{x} and \mathbf{y} are given below.

D.2.1 Puncture [3, 2]

The argument of \tilde{P} is

$$\mathbf{x} = (it^{1/2}, t) , \quad (\text{D.19})$$

while the prefactor is

$$\begin{aligned} \tilde{K}_{[3,2]} &= \frac{(-t; q)}{\{\prod_{i=1}^3 ((-t)^i; q)\} \{\prod_{j=1}^2 ((-t)^j; q)\}} \text{PE} \left[\frac{u(t^3 + t^5)}{1 - q^2} \right]_{u=-1} \\ &= \frac{(-t; q)}{\{\prod_{i=1}^3 ((-t)^i; q)\} \{\prod_{j=1}^2 ((-t)^j; q)\} \{(-t^3; q^2)(-t^5; q^2)\}} . \end{aligned} \quad (\text{D.20})$$

D.2.2 Puncture [2, 2, 1]

The argument of \tilde{P} is

$$\mathbf{y} = (iyt^{1/2}, iy^{-1}t^{1/2}) , \quad (\text{D.21})$$

while the prefactor is

$$\begin{aligned} \tilde{K}_{[2,2,1]} &= \frac{(-t; q)}{\{\prod_{i=1}^2 ((-t)^i; q)(-(-t)^i; q)\} \{(-t; q)\}} \times \\ &\quad \text{PE} \left[\left\{ \frac{u(t + t^2)}{1 - q} + \frac{ut^3}{1 - q^2} \right\} (y^2 + y^{-2}) \right]_{u=-1} . \end{aligned} \quad (\text{D.22})$$

D.3 A sphere with punctures [3, 3], [3, 2, 1] and [1, 1, 1, 1, 1, 1]

Without the twisting, in terms of chiral superfields, we have matter in the following representations of $\text{SU}(2) \times \text{SU}(6)$: $[1; 1, 0, 0, 0, 0]$, $[1; 0, 0, 0, 0, 1]$, $[0; 0, 1, 0, 0, 0]$ and $[0; 0, 0, 0, 1, 0]$.

The only possible \mathbb{Z}_2 action exchanges $[0, 1, 0, 0, 0]$ with $[0, 0, 0, 1, 0]$, and $[1, 0, 0, 0, 0]$ with $[0, 0, 0, 0, 1]$. Up to this point, we are still keeping all the fugacities of $\text{SU}(6)$, namely z_1, \dots, z_6 . Note the characters of the following representations

$$\begin{aligned} \chi_{[1,0,0,0,0]}^{A_5} &= \sum_{i=1}^6 z_i , & \chi_{[0,0,0,0,1]}^{A_5} &= \sum_{i=1}^6 z_i^{-1} , \\ \chi_{[0,1,0,0,0]}^{A_5} &= \sum_{1 \leq i < j \leq 6} z_i z_j , & \chi_{[0,0,0,1,0]}^{A_5} &= \sum_{1 \leq i < j \leq 6} z_i^{-1} z_j^{-1} . \end{aligned} \quad (\text{D.23})$$

The \mathbb{Z}_2 operation thus sends $z_i \rightarrow 1/z_i$. Next, we set

$$z_6 = z_1^{-1}, \quad z_5 = z_2^{-1}, \quad z_4 = z_3^{-1} . \quad (\text{D.24})$$

Hence, under the replacement (D.24) and taking into account of the doubling, we have

$$\begin{aligned} \chi_{[1]}^{A_1}(\mathbf{x}) \chi_{[1,0,0,0,0]}^{A_5}(\mathbf{z}) + \chi_{[1]}^{A_1}(\mathbf{x}) \chi_{[0,0,0,0,1]}^{A_5}(\mathbf{z}) &\rightarrow (x^2 + x^{-2}) E_1(\mathbf{z}) , \\ \chi_{[0,1,0,0,0]}^{A_5}(\mathbf{z}) + \chi_{[0,0,0,1,0]}^{A_5}(\mathbf{z}) &\rightarrow E_2(\mathbf{z}) + 3 . \end{aligned} \quad (\text{D.25})$$

The twisted index is

$$\tilde{\mathcal{I}}_{[3,3],[3,2,1],[1^6]} = \text{PE} \left[\frac{ut}{1-q} \{ (x^2 + x^{-2})E_1(\mathbf{z}) + (E_2(\mathbf{z}) + 3) \} \right]_{u=-1}. \quad (\text{D.26})$$

This can be written in the following form

$$\tilde{\mathcal{I}}_{[3,3],[3,2,1],[1^6]} = \frac{\tilde{K}_{[3,3]} \tilde{K}_{[3,2,1]} \tilde{K}_{\text{full}}}{\tilde{K}_{\rho}} \sum_{\lambda} \frac{\tilde{P}_{\lambda}(\mathbf{x}) \tilde{P}_{\lambda}(\mathbf{y}) \tilde{P}_{\lambda}(\mathbf{z})}{\tilde{P}_{\lambda}(t^{\rho})}, \quad (\text{D.27})$$

where \tilde{K}_{Λ} , \mathbf{x} and \mathbf{y} are given below.

D.3.1 Puncture $[3, 3]$

The argument of \tilde{P} is

$$\mathbf{x} = (xt, x^{-1}t, x), \quad (\text{D.28})$$

while the prefactor is

$$\tilde{K}_{[3,3]} = \frac{(-t; q)}{\prod_{i=1}^3 ((-t)^i; q) (-(-t)^i; q)} \text{PE} \left[(x^2 + x^{-2}) \left\{ \frac{t}{1-q} + \frac{ut^2}{1-q} + \frac{t^3}{1-q} \right\} \right]_{u=-1}. \quad (\text{D.29})$$

D.3.2 Puncture $[3, 2, 1]$

The argument of \tilde{P} is

$$\mathbf{y} = (it, t^{1/2}, i), \quad (\text{D.30})$$

while the prefactor is

$$\begin{aligned} \tilde{K}_{[3,2,1]} &= \frac{(-t; q)}{\{ \prod_{i=1}^3 ((-t)^i; q) \} \{ \prod_{j=1}^2 ((-t)^j; q) \} \{ (-t; q) \}} \times \\ &\quad \text{PE} \left[\frac{u(t^3 + t^5)}{1 - q^2} + \frac{t^4}{1 - q^2} + \frac{ut^3}{1 - q^2} \right]_{u=-1} \\ &= \frac{(-t; q)}{\{ \prod_{i=1}^3 ((-t)^i; q) \} \{ \prod_{j=1}^2 ((-t)^j; q) \} \{ (-t; q) \} \{ (-t^3; q^2)^2 (t^4; q^2) (-t^5; q^2) \}}. \end{aligned} \quad (\text{D.31})$$

D.4 A sphere with punctures $[4, 2]$, $[2, 2, 2]$ and $[1, 1, 1, 1, 1, 1]$

Without the twisting, in terms of chiral superfields, we have matter in the following representations of $\text{SU}(3) \times \text{SU}(6)$: $[1, 0; 1, 0, 0, 0, 0]$, $[0, 1; 0, 0, 0, 0, 1]$, $[0, 0; 0, 0, 1, 0, 0]$, see [30].

The only possible \mathbb{Z}_2 action exchanges $\text{SU}(6)$ representation $[1, 0, 0, 0, 0]$ with $[0, 0, 0, 0, 1]$, and $\text{SU}(3)$ representation $[1, 0]$ with $[0, 1]$. As before, the \mathbb{Z}_2 operation thus sends $z_i \rightarrow 1/z_i$ (where z_1, \dots, z_6 are $\text{SU}(6)$ fugacities) and $y_i \rightarrow 1/y_i$ (where y_1, y_2, y_3 are $\text{SU}(3)$ fugacities). Next, we set

$$z_4 = z_1^{-1}, \quad z_5 = z_2^{-1}, \quad z_6 = z_3^{-1}. \quad (\text{D.32})$$

Similarly for $SU(3)$, we set

$$y_1 = y, \quad y_2 = 1, \quad y_3^{-1} = y. \quad (D.33)$$

The twisted character for the chiral fields in $[1, 0; 1, 0, 0, 0, 0] + [0, 1; 0, 0, 0, 0, 1]$ of $SU(3) \times SU(6)$ can be computed using the replacements (D.32), (D.33) and taking into account of the doubling; we have

$$\chi_{[1,0]}^{A_2}(\mathbf{y})\chi_{[1,0,0,0,0]}^{A_5}(\mathbf{z}) + \chi_{[0,1]}^{A_2}(\mathbf{y})\chi_{[0,0,0,0,1]}^{A_5}(\mathbf{z}) \rightarrow (y^2 + 1 + y^{-2})E_1(\mathbf{z}). \quad (D.34)$$

Let us now consider the chiral fields in $[0, 0, 1, 0, 0]$ of $SU(6)$. Under the replacement (D.32) (without any doubling at this stage), we have

$$\chi_{[0,0,1,0,0]}^{A_5}(\mathbf{z}) \rightarrow 2 \sum_{i=1}^3 (z_i + z_i^{-1}) + \sum_{\epsilon_i=\pm 1} z_1^{\epsilon_1} z_2^{\epsilon_2} z_3^{\epsilon_3} = 2E_1(z_{1/2}) + E_3(\mathbf{z}). \quad (D.35)$$

Since the representation $[0, 0, 1, 0, 0]$ is real, there is no doubling for the second term $E_3(\mathbf{z})$ of (D.35). However, there is still a doubling for the first term $2E_1(z_{1/2})$ of (D.35); this leads to $E_1(\mathbf{z})$.

Thus, the twisted index is

$$\begin{aligned} \tilde{\mathcal{I}}_{[4,2],[2,2,2],[1^6]} = \text{PE} \left[\frac{t}{1-q^2} (y^2 + y^{-2} + 1) E_1(\mathbf{z}) + \frac{t}{1-q^2} E_1(\mathbf{z}) \right. \\ \left. + \frac{ut^{1/2}}{1-q} E_3(\mathbf{z}) \right]_{u=-1}. \end{aligned} \quad (D.36)$$

This can be written in the following form

$$\tilde{\mathcal{I}}_{[4,2],[2,2,2],[1^6]} = \frac{\tilde{K}_{[4,2]} \tilde{K}_{[2,2,2]} \tilde{K}_{\text{full}}}{\tilde{K}_{\rho}} \sum_{\lambda} \frac{\tilde{P}_{\lambda}(\mathbf{x}) \tilde{P}_{\lambda}(\mathbf{y}) \tilde{P}_{\lambda}(\mathbf{z})}{\tilde{P}_{\lambda}(t^{\rho})}, \quad (D.37)$$

where \tilde{K}_{Λ} , \mathbf{x} and \mathbf{y} are given below.

D.4.1 Puncture $[4, 2]$

The argument of \tilde{P} is

$$\mathbf{x} = (t^{3/2}, t^{1/2}, t^{1/2}) \quad (D.38)$$

while the prefactor is

$$\tilde{K}_{[4,2]} = (-t; q) \frac{1}{(t^4; q^2)(t^6; q^2)} \prod_{i=1}^4 \frac{1}{((-t)^i; q)} \prod_{i=1}^2 \frac{1}{((-t)^i; q)}. \quad (D.39)$$

D.4.2 Puncture [2, 2, 2]

The argument of \tilde{P} is

$$\mathbf{y} = (yt^{1/2}, y^{-1}t^{1/2}, -t^{1/2}) \quad (\text{D.40})$$

while the prefactor is

$$\begin{aligned} \tilde{K}_{[2,2,2]}(\mathbf{y}) = & \text{PE} \left[\left(\frac{ut}{1-q} + \frac{t^2}{1-q} + \frac{t^2}{1-q^2} + \frac{t^2}{1-q^4} \right) E_1(\mathbf{y}) \right]_{u=-1} \\ & \times \frac{1}{(-t; q)^2 (t^2; q)^2 (t; q) (-t^2; q)} . \end{aligned} \quad (\text{D.41})$$

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